The Riemann-Hilbert problem, orthogonal polynomials, and random matrix theory

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I give a short introduction to the Riemann-Hilbert problem and its connection to orthogonal polynomials and the theory of random matrices.

I. INTRODUCTION

At the Paris conference of the International Congress for Mathematicians in 1900 and later in Sorbonne, David Hilbert put forth 23 unresolved problems in mathematics.[4] Hilberts problems had a big impact on research done during the 20^{th} century until today. Hilberts twenty-first problem concerned the theory of *Fuchsian systems*,[2] i.e. $N \times N$ systems of ordinary linear differential equations of the form

$$\frac{d\psi(\lambda)}{d\lambda} = A(\lambda)\psi(\lambda) \tag{1}$$

where $\psi(\lambda)$ is a matrix-valued function and $A(\lambda)$ is a coefficient matrix whose $N \times N$ components are rational functions of λ with simple poles. The fundamental solution (Greens matrix) at some point can be continued analytically to the punctured Riemann sphere (punctured at the poles of $A(\lambda)$). The fundamental solution, after a closed contour γ on the punctured Riemann sphere, only depend on the homotopy class of the contour. Therefore, the fundamental solutions of a Fuchsian system define a matrix representation of the fundamental group of the punctured Riemann sphere. This representation is called the *monodromy group*. The original question of Hilbert was the inverse: can we always find a Fuchsian system for a given monodromy group?

The subsequent developments resulted in the year 1957 in a negative answer to Hilberts twenty-first problem in its full generality. However, a deep analysis of solvability conditions produced many new analytical tools and results.[2]

Later developments in mathematics and mathematical physics put Hilberts twenty-first problem in a much broader context. It turned out that many problems in pure or applied mathematics can be formulated in similar ways as Hilberts original problem. The classical example is the *Wiener-Hopf* method to solve partial differential equations. In a similar context, applications in the theory of *integrable systems* turned out to be particularly fruitful.[1] Those generalizations of Hilberts twenty-first problem was named *Riemann-Hilbert problem*. The solutions to many nonlinear integrable differential and difference equation may be formulated as a solution to a (generalized) Riemann-Hilbert problem.

Random matrix theory (RMT) originally comes from a completely different branch of physics. Random matrices where first intensively studied in connection with nuclear physics by Wigner in the 1950s. The underlying assumption is that quantum mechanical spectra of physical systems with many degrees of freedom may behave (at an appropriate scale) like the eigenvalues of a matrix with random elements. One approach in theoretical physics was to view random matrix statistics as the quantum analog to classical *chaotic* (NB: in contrast to integrable) dynamical systems.[3] More recent research on random matrix theory has led to various applications and connections to other fields of mathematics. Hitherto unexpected links where made e.g. to number theory (distribution of the zeros of the Riemann zeta function)[9] or to the problem of perfect matching of a graph.[11]

More recently, the research on random matrix ensembles have revealed further unexpected and possibly deep connections to the theory of integrable system. Aspects of this connection were discussed at the CRM conference in Montreal in summer 2005.[8] For example, gap probabilities in the Unitary ensemble may be formulated (in the appropriate scaling limit $N \to \infty$) in terms of solutions to integrable PDEs or *Painlevé type* of nonlinear ODEs. As another example, the finite-N probabilities may be mapped to solutions of *discrete* integrable models like the *Toda lattice*.[10]

II. THE RIEMANN-HILBERT PROBLEM

The Riemann-Hilbert problem (RHP) in its generalized form can be formulated in the following way:
Let Γ be an oriented contour or a set of oriented contours in the complex plane. The contours may be simple or disconnected, or even self intersecting. The orientation defines $+$ and $-$ sides of the contour in the usual way. Suppose further that we have a map G from Γ into the set of $N \times N$ invertible matrices. The Riemann-Hilbert problem determined by the pair (Γ, G) consists in finding the matrix-valued function $Y(\lambda)$ with the following properties:

- $Y(\lambda)$ is analytic for $\lambda \in \mathbb{C} \setminus \Gamma$
- $Y_+(\lambda) = Y_-(\lambda)G(\lambda)$ for $\lambda \in \Gamma$
  with $Y_\pm(\lambda) = \lim_{\lambda_0 \to \lambda^\pm} Y(\lambda_0)$
  and $\lambda^\pm$ is on the $\pm$ side of Γ.
- $Y(\lambda) \to 1$ as $\lambda \to \infty$.

The last condition on the limit has to be stated more precisely for a particular application. In connection with random matrices, we often encounter different boundary conditions. The crucial requirement is the "jump condition" along the contour Γ. An appropriate limit (e.g. at infinity) reduce the number of solutions.

The simplest example of an explicit solution to the Riemann-Hilbert problem is the case $N = 1$. Taking the logarithm, the jump-condition becomes additive and we clearly have

$$\log Y(z) = \int_\Gamma \frac{\log G(\lambda)}{\lambda - z} \frac{d\lambda}{2\pi i} \,. \tag{2}$$

There is a lot to say about the uniqueness of solution (2) or its generalization to the matrix case. As indicated, these questions are directly related to Hilberts 21^{st} problem. However, I will not go into these details here but instead discuss one application in random matrix theory.

III. A RHP FOR ORTHOGONAL POLYNOMIALS

Orthogonal polynomials play a central role in the recent developments of random matrix theory. In the following I will explain one basic result which is repeatedly used: orthogonal polynomials appear as a solution to a RHP with $N = 2$.

Given the following family of RHPs (Γ, G) with $\Gamma = \mathbb{R}$ and n an integer:

- Y_n analytic in $\mathbb{C} \setminus \mathbb{R}$
- $Y_{n+} = Y_{n-} G$ for $z \in \mathbb{R}$
- $Y_n \to \left(1 + O(z^{-1})\right) z^{n\sigma_3}$ as $z \to \infty$

where we denote the limit

$$z^{n\sigma_3} = \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix} \tag{3}$$

and the jump-matrix is given by

$$G(x) = \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix} . \tag{4}$$

The function $w(x)$ is some weight function on the real axis.

The unique solution to the above RHP can be written in terms of orthogonal polynomials of w:

$$Y_n = \begin{pmatrix} \pi_n & C\pi_n \\ \gamma_{n-1}\pi_{n-1} & \gamma_{n-1} C\pi_{n-1} \end{pmatrix} \tag{5}$$

where π_n are the monic orthogonal polynomials of w, $\int \pi_n(x)\pi_m(x)w(x)\,dx = -2\pi i\,\delta_{nm}\gamma_n^{-1}$, and C is the Cauchy transform:

$$C\pi_n(z) = \int_\Gamma \frac{\pi_n(s)w(s)}{s - z} \frac{ds}{2\pi i} \,. \tag{6}$$

Similar to the case of result (2) for the $N = 1$ RHP, above solution for $N = 2$ can be proven formally by elementary means. I will not repeat the derivation here but refer to introductory literature [5,12] for details. However, I will briefly comment on the uniqueness of the solution: since the jump-matrix has $\det G = 1$, we see that $\det Y$ must be analytic in the whole complex plane. Furthermore, $\det Y \to 1$ as $z \to \infty$ and we clearly have $\det Y = 1$ in the entire complex plane, by the *Liouville theorem* (every bound analytic function is a constant). Therefore, Y is invertible everywhere. Suppose $\tilde{Y}$ is another solution to the RHP. Then we can define $X = \tilde{Y} Y^{-1}$. Along Γ we have

$$X_- = \tilde{Y}_- Y_-^{-1} = \tilde{Y}_+ G\, G^{-1} Y_+^{-1} = \tilde{Y}_+ Y_+^{-1} \tag{7}$$

and hence the matrix X is analytic across the Γ. Since $X \to 1$ as $z \to \infty$, we again have $X = 1$ in the entire complex plane and hence $Y = \tilde{Y}$.

IV. ORTHOGONAL POLYNOMIALS AND RHP IN RANDOM MATRIX THEORY

Let me briefly remind some basic results concerning orthogonal polynomials. Let $w(x)$ be a weight function on the real axis and we will write $d\mu(x) = w(x)dx$. We require all moments to be finite, i.e. $\int x^n d\mu(x) < \infty$. The orthonormal polynomials for $w(x)$ are defined as

$$\int p_n(x) p_m(x)\, d\mu(x) = \delta_{nm} \,. \tag{8}$$

p_n are polynomials of degree n

$$p_n(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 \,. \tag{9}$$

The polynomials p_n are unique and can be constructed by the *Gram-Schmidt* orthogonalization procedure.

There is a very important *three term recurrence relation* for orthogonal polynomials. The polynomial $xp_n(x)$ is clearly of degree $n+1$. Further, it is easy to see that it must be a superposition of maximally three polynomials:

$$xp_n(x) = b_{n+1} p_{n+1}(x) + c_n p_n(x) + b_n p_{n-1}(x) \,. \tag{10}$$

The coefficients are given by

$$c_n = \int x p_n(x)^2 \, d\mu(x) \,, \tag{11a}$$

$$b_n = \int x p_n(x) p_{n-1}(x)\, d\mu(x), \text{ for } n > 0 \,, \tag{11b}$$

$$b_0 = 0 \,. \tag{11c}$$

Using the three term recurrence relation (10), we immediately find the *Christoffel-Darboux formula* for orthogonal polynomials:

$$\begin{aligned} &(x - y) \sum_{j=0}^{N-1} p_j(x) p_j(y) \\ &\quad = b_N \left(p_N(x) p_{N-1}(y) - p_{N-1}(x) p_N(y)\right) . \end{aligned} \tag{12}$$

Let me now turn to random matrix theory. For simplicity, I will consider only the unitary ensemble with the following measure in the space of $N \times N$ complex hermitian matrices

$$d\mu(M) = Z_N^{-1} e^{-trV(M)} dM \tag{13}$$

where $V(x)$ is a polynomial of even degree and

$$dM = \Pi_{j=1}^N dM_{jj} \Pi_{i \neq j}^N dReM_{ij} dImM_{ij} \,. \tag{14}$$

The probability measure (13) is invariant under unitary transformations $M \to U^\dagger M U$ and therefore we can diagonalize and integrate out the “angle” variables to get a measure on the eigenvalues λ_j

$$\begin{aligned} d\mu(\lambda) &= Z_N^{-1} \, \Pi_{i<j} (\lambda_i - \lambda_j)^2 \, e^{-\sum_k V(\lambda_k)} \, \Pi_l d\lambda_l \\ &= P_N(\lambda) \, d\lambda \,. \end{aligned} \tag{15}$$

Further, it is clear that the term $\Pi_{i<j}(\lambda_i - \lambda_j)^2$ can be written as a *van der Monde* determinant

$$\Pi_{i<j}(\lambda_i - \lambda_j)^2 = \left|\lambda_j^{i-1}\right|^2 \,. \tag{16}$$

The orthogonal polynomials $\{p_n\}_{n=0 \dots N}$ for the weight function $w(x) = e^{-V(x)}$ form a complete set of basis vectors. Therefore, we can add and subtract rows in the van der Monde determinant to write the measure in terms of orthogonal polynomials:

$$P_N(\lambda) = \frac{1}{N!} \, |\det M|^2 \tag{17}$$

where $M_{ij} = \phi_{i-1}(\lambda_j)$ and

$$\phi_n(\lambda) = p_n(\lambda) \, e^{-V(\lambda)/2} \tag{18}$$

Now, we have

$$\begin{aligned} |\det M|^2 &= \det M^T M = \det \sum_{n=0}^{N-1} \phi_n(\lambda_i) \phi_n(\lambda_j) \\ &= \det[K_N(x_i, x_j)]_{1 \leq i,j \leq N} \,. \end{aligned} \tag{19}$$

Due to orthogonality of ϕ_n, this kernel has the nice properties

$$\begin{aligned} &\int K_N(x,x) \, dx = N \,, \\ &\int K_N(x,y) K_N(y,z) \, dy = K_N(x,z) \,. \end{aligned} \tag{20}$$

Furthermore, Christoffel-Darboux (12) allows us to write the kernel as

$$\begin{aligned} K_N(x,y) &= b_N \, \frac{\phi_N(x)\phi_{N-1}(y) - \phi_N(y)\phi_{N-1}(x)}{x-y} \\ K_N(x,x) &= b_N \left[\phi_N'(x)\phi_{N-1}(x) - \phi_N(x)\phi_{N-1}'(x)\right] . \end{aligned} \tag{21}$$

The m-point correlation functions are defined as

$$R_N^{(m)}(\lambda_1, \dots, \lambda_m) = \frac{N!}{(N-m)!} \int P_N(\lambda) \, \Pi_{j=m+1}^N d\lambda_j \,. \tag{22}$$

Using (20), one shows that

$$R_N^{(m)}(\lambda_1, \dots, \lambda_m) = \det[K_N(\lambda_i, \lambda_j)]_{1 \leq i,j \leq m} \,. \tag{23}$$

Now it becomes clearer why the Riemann-Hilbert formulation of the problem may be useful. Going back to the RHP solution for orthogonal polynomials, (5), we see that the kernel can be written as

$$K_N(x,y) = \frac{e^{-(V(x)+V(y))/2}}{2\pi i(x-y)} \; (0 \;\, 1) \, Y_N^{-1}(y) Y_N(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{24}$$

Solving the Riemann-Hilbert problem (4) will give us the kernel for the matrix model. Particularly interesting are universality questions in the limit of large random matrices, $N \to \infty$. Using this approach, the universality question of K_N has been answered for an arbitrary polynomial potential V of degree $2m$. Upon appropriate rescaling $\lambda \to N^{\frac{1}{2m}} \lambda$, one obtains the sine-kernel in the scaling limit $N \to \infty$ [5]:

$$K_N \to \frac{\sin \pi(x-y)}{\pi(x-y)} \,. \tag{25}$$

Universality questions in RMT have motivated much work on the large N asymptotics of orthogonal polynomials. Considerable progress was made by Kuijlaars *et al.* [7] and others using the the Riemann-Hilbert approach. Note also that the $z \to \infty$ asymptotic solutions of the RHP give valuable information about the orthogonal polynomials. From (5) we find for example

$$[Y_n]_{12} = -\frac{k_n}{z^{n+1}} + O\left(\frac{1}{z^{n+2}}\right) \tag{26}$$

where $k_n = \int \pi_n^2(x) \, d\mu(x)$.

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