Diploma Work

Stability of the soliton in the broken $O(3)$ nonlinear σ-model in $2+1$ dimensions

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Résumé

Dans ce travail de diplôme nous étudions un soliton dans le contexte de la théorie des champs quantiques. Le modèle considéré est le modèle σ non-linéaire à symétrie $O(3)$ en 3 dimensions avec un terme de masse qui brise la symétrie. La propriété de stabilité du soliton “hérisson” circulairement symétrique avec indice topologique 1 est analysée classiquement ainsi que par les méthodes de la mécanique quantique. En ce qui concerne la théorie des champs classique, un code numérique qui intègre les équations différentielles aux dérivées partielles du problème a été développé. Le résultat nous montre que le soliton se rétrécit à une configuration singulière de champ et ceci dans un temps fini. L'approche de la mécanique quantique utilise la méthode des coordonnées collectives dans une variable de dilatation. Il s'avère que le soliton, qui est classiquement instable, est stabilisé par des effets quantiques. Le spectre d'énergie est celui d'un oscillateur harmonique. Si les conclusions sont correctes et si les “solitons quantiques” existent, ceci pourrait avoir des applications intéressantes et nouvelles en cosmologie.

Abstract

In this diploma work we discuss a soliton in the context of quantum field theory. The model considered is the $O(3)$ nonlinear σ-model in 3 dimensions with a symmetry breaking mass term. The stability properties of the circularly symmetric “hedgehog” soliton with topological number 1 is analyzed both classically and quantum mechanically. As far as classical field mechanics is concerned, a numerical code has been developed which integrates the partial differential equation of the problem. The result is that the soliton shrinks to a singular field configuration in finite time. The quantum mechanical calculations use the collective coordinate approach in a dilatation variable. The classically instable soliton is found to be stabilized through quantum effects. The spectrum of the quantum soliton is that of a harmonic oscillator. If the conclusions are correct and “quantum solitons” exist, this could have interesting and novel applications in cosmology.

Contents

1 Introduction

The soliton was discovered more than 150 years ago by John Scott Russel. The Scottish engineer described for the first time a peculiar phenomenon he had observed in the canals of Edinburgh: a wave that does not change its shape while propagating. The importance of Russel's observation was only recognized years after his death by the scientific community. Mathematicians found examples of nonlinear differential equation which have non-dispersive solutions, they were finally identified with what Scott Russel had called "solitary waves". Today, solitons are of great importance not only to mathematics, but also to applied sciences and engineering. Be it hydrodynamics, optics, electrical engineering, condensed matter, elementary particle and nuclear physics or cosmology, there is an impressive number of applications in various fields.

In the present work, a specific soliton is discussed in the context of elementary particle physics. Quantum field theory is the framework which is believed to govern the basic building blocks of nature. Solitons in quantum field theory are interpreted as extended particles, opposed to the ordinary, point-like particles. The soliton discussed here turns out to decay into a singular field configuration from the point of view of classical mechanics. However, classical instability does not necessarily mean physical decay. In physics, there are numerous examples of classically unstable configurations. For example the hydrogen atom cannot exist according to classical physics: an electron orbiting a charged nucleus will radiate its energy away and finally fall on top of the nucleus. However, quantum effects prevent this to happen. We conjecture that the same mechanism will stabilize our soliton. We apply a collective coordinate quantization to our model. And indeed, we find a quantum stabilized soliton.

This work is organized in the following way: In the second part of this section, we will give a brief introduction to solitons and some general information about nonlinear σ-models. The purpose is to sketch the background and the applications of this family of models. It is thought to address non-specialists in this field. In section 2, we present a classical analysis of our model, considering first the massless and then the massive case. In the third section we expose a quantization procedure and we show that the classically unstable soliton can be stabilized by quantum mechanics. In a series of appendices we give supplementary information on the methods used and show important theorems.

1.1 Some background information

In this section I want to give some general motivation on the subject. The intention is to put the investigations of this work into a slightly larger context.

1.1.1 What is a soliton ?

This work is about a soliton. I want to give here a very brief introduction to solitons in field theory. The intuitive approach is sufficient for our purpose. For a more concise introduction see Rajaraman [1]. For review of the history and applications of solitons see [2].

Let us for simplicity take the case of a scalar field $\phi(x,t)$. Suppose furthermore that the physical quantity *energy density* $\epsilon = \epsilon(\phi, \partial_t\phi, \partial_x\phi)$ can be associated to the field. For a field configuration to make physical sense, its total energy should stay finite and bound from below:

$$\begin{aligned} E(t) &= \int_{space} \epsilon(\phi(x,t))dx \\ 0 &\leq E(t) < \infty,\ \forall t \end{aligned} \tag{1.1}$$

The evolution of ϕ is given by a partial differential equation (PDE), derived from minimization of some action functional of the field $S[\phi] = \int dt\, L[\phi, t]$:

$$\frac{\delta S[\phi]}{\delta\phi} = 0 \tag{1.2}$$

A *solitary wave* is a nonsingular solution of this differential equation with localized energy density. More precisely

$$\begin{aligned} \epsilon(x,t) &= f(x - vt) < \infty \\ f(\xi) &\to 0 \text{ as } |\xi| \to \infty \end{aligned} \tag{1.3}$$

In other words, a solitary wave is a “lump of energy” which moves undistorted in shape.

Let $\epsilon_0(x,t)$ be the energy density of a solitary wave. It is a *soliton* if the equation of motion admits solutions with energy densities of the following form:

$$\begin{aligned} \epsilon(x,t) &\to \sum_{i=1}^{N} \epsilon_0(x - a_i - v_i t) \text{ as } t \to -\infty \\ \epsilon(x,t) &\to \sum_{i=1}^{N} \epsilon_0(x - b_i - v_i t) \text{ as } t \to +\infty \end{aligned} \tag{1.4}$$

where a_i, b_i and v_i are constants. In other words, a solitary wave is a soliton if the equation of motion admits widely separated multiple solitary waves which asymptotically restore their shape and velocity after collision.[1]

It is most interesting that solitary waves even exist for nonlinear equations. An example of a linear equation without dispersion is

$$(\partial_t^2 - \partial_x^2)\phi(x,t) = 0 \tag{1.5}$$

It has the solution $\phi(x,t) = \sin(k(x-t))$ for any $k \in \mathbb{R}$. Due to linearity, we can construct localized wave packets which are clearly solitons. These solitons are uninteresting in our context, precisely because there is a linear superposition principle. Physically speaking, the solitons do not interact with each other in this model.

However, as soon as the equation gets slightly more complicated, it is more difficult to find solitons. Take for example the relativistic *Klein-Gordon* equation:

$$(\partial_t^2 - \partial_x^2 - 1)\phi(x,t) = 0 \tag{1.6}$$

A solution is given by $\phi(x,t) = \sin(k(x - v(k)t)))$ with $v(k) = \sqrt{1+k^2}/k$ for any $k \in \mathbb{R}$. We can still construct localized wave packets, but the different waves move with different velocities. The packet spreads with time. Equation (1.6) is said to have dispersion.

Non-trivial equations which have soliton solutions must necessarily be nonlinear. In a sense, the nonlinear effects and the dispersion may cancel and a stable solitary wave results. As a last example let me give the so-called *"Sine-Gordon"* equation:

$$(\partial_t^2 - \partial_x^2)\phi - \lambda^{-1}\sin(\lambda\,\phi) = 0 \tag{1.7}$$

where λ is a real parameter. This equation has solitary waves given by $\phi(x,t) = \lambda^{-1} 4 \arctan(e^x)$ and it even admits soliton solutions which scatter and restore their shape afterwards.

A very general feature of solitons is that they are *non-perturbative*. As we see in our example of the Sine-Gordon equation, there is the parameter λ which determines somehow the degree of nonlinearity of the equation. If λ is sufficiently small, we could try expand the sine in (1.7) and we arrive at the linear *Klein*-Gordon equation (1.6) to order λ^2. It is interesting to see what

[1] These definitions are adopted from [1]. They ban in fact the majority of known finite energy solutions from the list of solitons. However, it is common in the literature to call all solitary waves solitons and I will adopt this practice after the introduction. It is still worth knowing the difference as it allows one to be more precise when desired.

happens to our soliton in this limit: it diverges. This is a observation which is true for all non-trivial solitons: they are not seen by perturbation theory. This is of crucial importance to quantum field theory, where perturbation theory is often the only tool available.

I have used the term classical stability. Let me define it more precisely: Let $\phi_c(x,t)$ be a solitary solution of a field theory. Consider the perturbed field $\phi(x,t) = \phi_c(x,t) + \eta(x)e^{i\omega t}$, where η is small and subject to the boundary condition $\eta \to 0$ as $|x| \to \infty$. We can now write a linearized equation of motion for $\eta(x)$. A solitary solution is called *classically stable* if this linearized equation has only solutions for real ω. This simply means that the solution should be stable in the sense that small perturbations do not grow exponentially in time.

One last remark concerns the physical interpretation of soliton solutions discussed so far. In quantum field theory, classically stable solitary waves are interpreted as extended particles ("classical lumps" Coleman [3], as opposed to point-like particles). If the model admits multi-solitary solutions or even solitons, these solutions are then interpreted as a system of interacting particles. Solitons are treated like classical objects, and physical quantities like energy etc. are calculated in a semi-classical approach [4, 5, 6, 1]. Form this point of view, classical stability is an essential requirement: the procedure of quantization introduces small perturbations to the solutions and they need to be classically stable in order to perform quantization consistently.

However, if the soliton is classically unstable there is no general method to treat it. It has been proposed by some authors that quantum effects can stabilize the soliton and give rise to so called "quantum solitons". I will discuss this possibility in detail in section 3.

1.1.2 Nonlinear σ-models

The non-linear σ-model I am considering in this work is defined by the following action:[1]

$$S = g \int d^3x \, \frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi_a - m^2 (1 - \phi_3) \; , \; \phi_a \phi_a = 1, \, a = 1 \ldots 3 \tag{1.8}$$

where g and m are real parameters. More eloquently, $\boldsymbol{\phi}(\mathbf{x}, t)$ is a 3-component real scalar field in two space and one time dimensions. $\boldsymbol{\phi}$ is furthermore

[1] The notation is explained in section 1.2.

constrained to live on a sphere S^2 with radius 1. In general, a nonlinear σ-model is a scalar field theory of which the target space[1] is a curved manifold. The model (1.8) is called $O(3)$ model, because its target manifold is invariant under $O(3)$ rotations of the field $\boldsymbol{\phi}$. It is called *broken* because the m^2 term breaks this rotational invariance of the action.

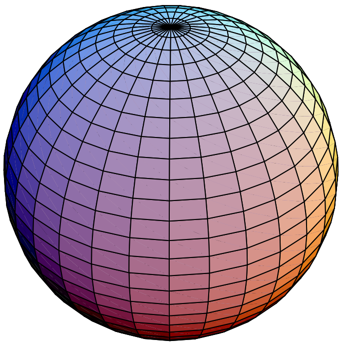

Figure 1.1: Target space of a simple non-linear σ-model, the one considered here.

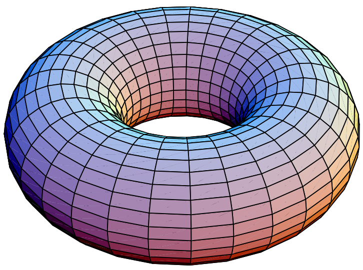

Figure 1.2: Target space of a more complicated non-linear σ-model.

[1] The target space is the space where $\boldsymbol{\phi}$ lives.

A very important nonlinear σ-model in nuclear and subnuclear physics is the *Skyrme model* [7, 8]. Its target manifold is $U(3)$ or more generally $U(N)$ which is a matrix space and not as easily visualized as our $O(3)$ model. The Skyrme model is believed to be an effective action for quantum chromodynamics (QCD), i.e. a low energy description of quark dynamics. One difficulty of QCD is that there are no small parameters to do a perturbational expansion. However, it was suggested by 't Hooft and then by Witten [9, 10] to expand the theory in the inverse of the number of colors, $1/N_c$. In this language, a baryon[1] which is composed out of N_c quarks, sees his mass grow with the expansion parameter $1/N_c$. We have seen before that this is a typical feature of a soliton. This and other theoretical reasons led to extensive search for the effective theory of QCD, possessing soliton solutions which could be interpreted as baryons. The Skyrme model is such a candidate, its solitons are called *skyrmions*. In this context, similar models to our $O(3)$ model are discussed in the literature as a simplified versions of the Skyrme model, their solitons are generally called *baby skyrmions* [11].

Another field of research where nonlinear σ-model find application is condensed matter physics. There, the spin of atoms on a lattice is considered. In the continuum limit, such a model is an $O(3)$ nonlinear σ-model. This approach is very important for the description and understanding of supraconductivity and antiferromagnets [12, 13].

Solitons find also applications in cosmology. Among others, the pressing questions of this field of research are the cosmological constant (hierarchy) problem or the puzzle of dark matter. In this context, there has been considerable interest in the study of extra dimensions and the brane world scenario in recent years (see e.g. [14] and references therein). One idea is that our 4 dimensional world is in fact the interior of a domain wall. This domain wall could be formed by a soliton living in higher dimensions. Other types of solitons in cosmology are superconducting cosmic strings or vortons [15].

[1] Proton, Neutron etc.

1.2 Notational conventions

In this work I will use the so-called "Einstein sum convention" about indices, i.e. repeated Greek indices are summed over from 0 to 2 and Arab indices from 1 to 3, i.e

$$\partial_\mu \phi_a \partial^\mu \phi_a \equiv \sum_{a=1}^{3} \sum_{\mu=0}^{2} \partial_\mu \phi_a \partial^\mu \phi_a$$

The metric is taken to be $g^{\mu\nu} \equiv diag(1, -1, -1)$. Thus,

$$x_\nu x^\nu = g^{\mu\nu} x_\mu x_\nu = x_0^2 - x_1^2 - x_2^2$$

Bold symbols are considered to be N component quantities with Arab indices, e.g.

$$\mathbf{x} = (x_1, x_2)$$

or

$$\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)$$

The symbol $\cdot$ is used as the euclidian scalar product:

$$\mathbf{x}^2 \equiv \mathbf{x} \cdot \mathbf{x} \equiv x_a x_a$$

The symbol $\wedge$ is used as the wedge product:

$$(\boldsymbol{\phi} \wedge \boldsymbol{\phi})_a = \epsilon^{abc} \phi_b \phi_c$$

ϵ is the completely antisymmetric tensor with 2 or 3 indices: $\epsilon^{123} = 1$ and $\epsilon_{12} = 1$.

If not otherwise stated, we use natural units where $\hbar = c = 1$. The energy is given in GeV.

$\dot{f} = \frac{df}{dt}$, $f' = \frac{df}{dr}$ where $r = \sqrt{x_1^2 + x_2^2}$.

1.3 Acknowledgements

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2 Classical discussion of the model

As we already stated in the introduction, the action of our model is given by:

$$S = g \int d^3x \, \frac{1}{2}\partial_\mu \phi_a \partial^\mu \phi_a - m^2(1-\phi_3) \ , \ \phi_a\phi_a = 1, \, a = 1\dots 3 \tag{2.1}$$

The energy functional corresponding to this action is:

$$H[\phi] = g \int d^2x \, \frac{1}{2}(\, (\partial_t \phi_a)^2 + (\boldsymbol{\nabla}\phi_a)^2) + m^2(1-\phi_3) \tag{2.2}$$

The field $\boldsymbol{\phi}$ is dimensionless, g and m are parameters with dimension of energy.

In the case $m^2 = 0$, the action (2.1) has $O(3)$ symmetry, i.e. we can rotate $\boldsymbol{\phi}$ without changing the action. In the language of QFT, this symmetry is spontaneously broken to $U(1)$ by the choice of a vacuum value $\boldsymbol{\phi}_{vac}$. If $m^2 \neq 0$, this symmetry is broken explicitly and the vacuum is $\boldsymbol{\phi}_{vac} = (0, 0, 1)$.

2.1 Topological charge

Without even looking at the equations of motion, we can readily classify the possible finite energy solutions by a topological argument. Due to the potential term m^2 in (2.2), a finite energy field configuration has to go to the vacuum at infinity:

$$\boldsymbol{\phi}(\mathbf{x}, t) \to (0, 0, 1) \text{ as } |\mathbf{x}| \to \infty, \forall t \tag{2.3}$$

The physical space where $\mathbf{x}$ takes values is thus undistinguishable at infinity. In other words, the original space $\mathbb{R}^2$ on which $\boldsymbol{\phi}$ is defined is reduced to $\mathbb{R}^2$ with all "points" at spatial infinity identified. This space is topologically equivalent to a sphere S^2_{phys}. On the other hand, the internal (target) space is the sphere S^2_{int} by definition. As a consequence, a field $\boldsymbol{\phi}$ respecting the boundary condition is a map $\boldsymbol{\phi} : S^2_{phys} \to S^2_{int}$, element of the second homotopy classes $\pi_2(S^2_{int}) \equiv \mathbb{Z}$. It is thus characterized by an integer winding number n. The homotopy or topological index of a field configuration $\boldsymbol{\phi}$ in a class n can be written explicitly as:[1]

$$n = \frac{1}{8\pi} \int_{S^2_{phys}} d^2x \, \epsilon^{abc}\epsilon_{ij} \, \phi_a \partial_i \phi_b \partial_j \phi_c \tag{2.4}$$

[1]For the critical reader this is explained in more detail in appendix A.

At this point it is important to note that (2.4) is conserved for any smooth transformation of the fields, in particular during time evolution. This statement is true for classical field theory and as long as the time evolution does not run into a singular field configuration. We will see later that this can happen in our model.

2.2 A lower bound on the energy

Still without using the equations of motion, we can show that the energy is bound from below for nonsingular fields. Consider the quantity

$$F_i^{a\pm} = \partial_i\phi_a \pm \epsilon^{abc}\epsilon_{ij}\,\phi_b\partial_j\phi_c \tag{2.5}$$

then, clearly

$$0 \leq F_i^{a\pm}F_i^{a\pm} = 2\,\partial_i\phi_a\partial_i\phi_a \mp 2\,\epsilon^{abc}\epsilon_{ij}\,\phi_a\partial_i\phi_b\partial_j\phi_c \tag{2.6}$$

or, equivalently

$$\partial_i\phi_a\partial_i\phi_a \geq \pm\epsilon^{abc}\epsilon_{ij}\,\phi_a\partial_i\phi_b\partial_j\phi_c \tag{2.7}$$

where we have used $\boldsymbol{\phi}\cdot\partial_i\boldsymbol{\phi} = 0$. If we integrate (2.7) over S^2_{phys}, we see the "gradient" term of the energy (2.2) appearing on the left hand side and the topological index on the right hand side. Kinetic ($\partial_t\boldsymbol{\phi}^2$) and potential ($m^2\ldots$) terms in the energy functional are positive definite. This allows us to set the following lower bound on the total energy:

$$H[\boldsymbol{\phi}] \geq 4\pi g|n| \tag{2.8}$$

where $\boldsymbol{\phi}$ has topological winding number n. This inequality was discovered by Belavin and Polyakov [12]. Similar bounds can be found in topological considerations of gauge theories and monopoles (see, e.g. [1]).

2.3 Equations of motion

The classical equations of motion for the fields ϕ_a can be derived using a Lagrange multiplier field $\lambda(\mathbf{x},t)$ to impose the constraint, and then by varying the action with respect to λ, and ϕ_a. This is done in [1, 16]. However, it seems more convenient here to resolve the constraint directly with the following substitution:

$$\boldsymbol{\phi} = \begin{pmatrix}\sin f\cos h\\ \sin f\sin h\\ \cos f\end{pmatrix} \tag{2.9}$$

where $f(\mathbf{x}, t)$ and $h(\mathbf{x}, t)$ are the new, unconstraint fields. They are simply the polar coordinates in internal space. The requirement of finite energy (2.3) imposes

$$f(\mathbf{x}, t) \to 2\pi N \text{ as } |\mathbf{x}| \to \infty \tag{2.10}$$

where N is an integer. However, these vacua are equivalent (i.e. they correspond to the same field $\boldsymbol{\phi}$) and we can set $N = 0$.

In this language, the action (2.1) is given by

$$S = g \int dt\, d^2x\, \frac{1}{2}(\partial_\mu f)^2 + \frac{1}{2}(\partial_\mu h)^2 \sin^2 f - m^2(1 - \cos f) \tag{2.11}$$

The residual $U(1)$ symmetry is $h \to h + \alpha$ and the corresponding Noether current is:

$$j_\mu = \partial_\mu h\ \sin^2 f \tag{2.12}$$

The topological charge (2.4) is

$$n = \frac{1}{4\pi} \int dx^2\, \epsilon_{ij}\, \partial_j h\, \partial_i (\cos f) \tag{2.13}$$

It is interesting to write (2.13) in polar coordinates (r, θ). Let us choose the origin $\mathbf{x_0}$ in such a way that $\sin(f(\mathbf{x_0}))$ vanishes. We can then have non-trivial dependency $h(\theta)$. After some integrations by parts, we get

$$n = \frac{1}{4\pi} \int dr\ h(r,\theta)\, \partial_r (\cos f)|_\theta^{\theta+2\pi} \tag{2.14}$$

Jumps of 2π are allowed for f and h. We thus have $h(r,\theta) - h(r,\theta + 2\pi) = 2\pi M(r)$, $M(r)$ integer, and $(\cos f)|_\theta^{\theta+2\pi} = 0$. We can now perform the integration and get

$$n = -M(\infty)\ \cos f|_{r=0}^{r=\infty}/2 = M(\infty)(\cos f(r = 0) - 1)/2 \tag{2.15}$$

The important point here is that we have to start with $f(r = 0) = \pi$ in order to obtain solutions with nontrivial topology. If this is fulfilled, the actual winding number n does only depend on the behavior of h. Hence, we can suppose that f goes continuously from π to 0 as r increases and that it stays in this interval.

Let us finally write the equations of motion:

$$\partial_\mu \partial^\mu f - \frac{\sin 2f}{2}(\partial_\mu h)^2 - m^2 \sin f = 0 \tag{2.16a}$$

$$\partial_\mu (\partial^\mu h\ \sin^2 f) = 0 \tag{2.16b}$$

Equation (2.16b) is simply the conservation equation for the current (2.12).

2.4 Massless case

The model with $m^2 = 0$ has static solitary solutions which are discussed in the literature in various contexts. I will consider only the special case of static, circularly symmetric solutions. The method for finding more general solutions is discussed in appendix B.

Consider the case of fields of the form $f = f(r)$ and $h = h(\theta, t)$. r and θ are polar coordinates. This is sometimes called the *hedgehog ansatz*. The equation of motion (2.16b) and the periodicity of h implies

$$h(\theta) = -n\,\theta + \theta_0 + \omega t \tag{2.17}$$

where n is an integer. (2.16) writes

$$f'' + \frac{1}{r} f' - \frac{\sin 2f}{2}\left(\frac{n^2}{r^2} - \omega^2\right) = 0 \tag{2.18}$$

In the following, we will only consider the case of a non-rotating soliton, i.e. we set $\omega = 0$. In this case (2.18) has an analytic solution:

$$f(r) = 2\arctan[\left(\frac{\lambda}{r}\right)^n] \tag{2.19}$$

where λ is an arbitrary constant. Other possible constants are fixed by the requirement $f \to 0$ as $r \to \infty$ and $f(r = 0) = \pi$. With this convention, the winding number (2.14) is exactly the n used here.

The energy densities of our hedgehog solutions are given by:

$$T^{00} = g\,(\frac{f'^2}{2} + \frac{\sin^2 f}{2r^2}) = g\,\frac{4\,n^2\left(\frac{\lambda}{r}\right)^{2\,n}}{r^2\left(1 + \left(\frac{\lambda}{r}\right)^{2\,n}\right)^2} \tag{2.20}$$

The energy densities of the solitons $n = 1$ and $n = 2$ are plotted in figure 2.1. The unit charge soliton has its energy density centered at the origin. Higher charged solutions form a ring around the origin. It is clear that the parameter λ measures the spatial extension of the soliton, we will thus call it the *size*.

In the following discussion, we will restrict ourself to the $n = 1$ solution.

2.4.1 Classical stability

Having found a static solution with $n = 1$, we should now examine its classical stability. To do this we apply a small radial perturbation to it (we will set $\theta_0 = 0$ from now on):

$$f(r, t) = 2\arctan\frac{\lambda}{r} + \eta_\omega(r)e^{i\omega t} \tag{2.21a}$$

$$h(\theta) = -\theta \tag{2.21b}$$

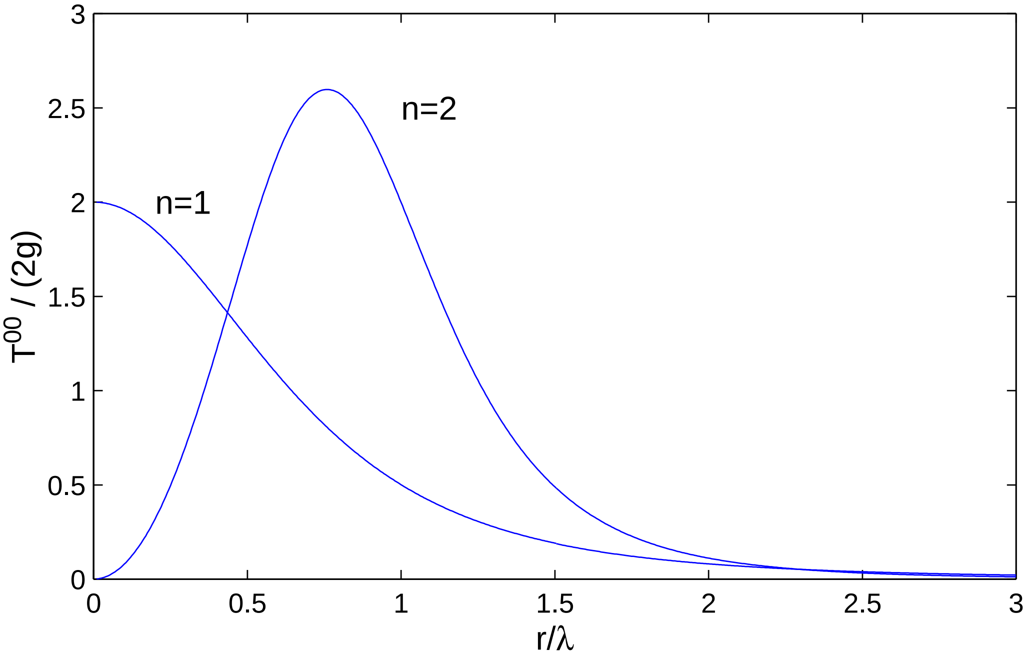

Figure 2.1: Radial energy density of the static $n = 1$ and $n = 2$ hedgehog solutions.

We put this into the equation of motion (2.16) and linearize for small η:

$$[-\partial_r^2 - \frac{1}{r}\partial_r + \frac{1}{r^2}(1 - \frac{8(\lambda/r)^2}{(1+(\lambda/r)^2)^2}) - \omega^2]\eta_\omega(r) = 0 \tag{2.22}$$

(2.22) is a Schrödinger eigenvalue problem (in 2 dimensions) with energy ω^2. The radial potential has a r^{-2} behavior at the origin and goes to zero at infinity. A a solution with $\omega = 0$ is given by

$$\eta_0(r) = \partial_\lambda(2\arctan\frac{\lambda}{r}) \propto \frac{r}{\lambda^2 + r^2} \tag{2.23}$$

This zero-mode has no node in the interval $(0, \infty)$. We know that a nodeless solution of a Schrödinger eigenvalue problem is the ground state with lowest energy. (2.23) is not normalizable, it is a so-called *half-bound state*. It does only vanish as r^{-1} at infinity. However, the theorem about nodes also holds for half-bound states. This tells us that there is no mode with $\omega^2 < 0$ and that the soliton is classically stable in the sense given in the introduction.

2.4.2 Dilatation invariance and marginal stability

(2.23) is a very unusual zero-mode which is rarely discussed in the literature. Normally, a zero-mode in soliton theory is due to a continuous symmetry of the action with respect to a field transformation $S[\phi] = S[e^{i\alpha T}\phi]$ where T is the generator of the symmetry and α some real number. $T\phi$ is then the corresponding zero-mode. For example translational invariance of a theory leads to an arbitrary position of the soliton and to a zero-mode.[1]

The arbitrary parameter λ in our soliton solution makes one think of some existing *dilatation symmetry* of our model. The transformation

$$x_\mu \to \alpha\, x_\mu \tag{2.24a}$$

$$\boldsymbol{\phi} \to \boldsymbol{\phi}/\sqrt{\alpha} \tag{2.24b}$$

leaves the action (2.1) invariant, but it violates the constraint on the norm of $\boldsymbol{\phi}$. On the other hand, the transformation

$$x_i \to \alpha\, x_i \tag{2.25}$$

leaves the potential energy in (2.2) $\int d^2x\,(\partial_i\phi_a)^2$ invariant, but it is not a true symmetry of the theory: the kinetic energy $\mathcal{T} = \int d^2x\,(\dot{\phi}_a)^2$ transforms as $\mathcal{T} \to \alpha^2\mathcal{T}$. Only time independent solutions have dilatation and even *conformal* symmetry in the sense that from one known static solution we can find another one through a conformal transformation on (x_1, x_2). However, this symmetry does not have an associated conserved Noether current.[2]

The fact that the zero-mode is non-normalizable makes it impossible for this mode to be excited. It would cost an infinite amount of energy. This is reminiscent of the pseudo dilatation symmetry.

The arbitrariness of its size leaves the soliton only *marginally stable*. Lattice simulations of this model in full $2+1$ dimensions have been performed [17, 18]. The results suggest that small perturbation of the $n = 1$ soliton will make its size shrink or expand, depending on the applied perturbation. The authors show that the shrinking occurs approximately linearly, $\lambda(t) = \lambda_0 + vt$ in (2.21a), if one assumes λ to be time independent at large distance from the soliton center. Such a cutoff is required to keep the kinetic energy finite.

[1] This symmetry, however, is absent here because we pinned the soliton at the origin by the hedgehog ansatz.

[2] In contrast, in $1+1$ dimensions the massless σ-model has true conformal symmetry.

2.5 Massive case

Let us now proceed to the model with $m^2 \neq 0$. No analytic solution is known in the massive case. Furthermore, a well known theorem due to *Derrick* [19] forbids the existence of static finite energy solutions.[1] Suppose now we place a static soliton in some $m^2 = 0$ world and then adiabatically raise m. What would happen ? The derivation of Derrick's theorem suggests that the soliton *shrinks* in some sense. On the other hand, the topological index (2.13) is conserved anyway, with or without mass. In short: even if it seems evident that, in the massive case, the soliton should decay, it is not entirely clear how it can do this by keeping its topology (and also its energy) invariant. We addressed this question in a lattice simulation, which is described bellow.

2.5.1 Asymptotic solutions

Let us briefly summarize. We made the hedgehog-ansatz for the field $\boldsymbol{\phi}$ with $n = 1$

$$\boldsymbol{\phi}(r, \theta, t) = \begin{pmatrix} \cos\theta \sin f(r,t) \\ \sin\theta \sin f(r,t) \\ \cos f(r,t) \end{pmatrix} \tag{2.26}$$

The action for $f(r,t)$ is:

$$S[f] = 2\pi g \int dt\, r dr \left(\frac{1}{2}\dot{f}^2 - \frac{1}{2}f'^2 - \frac{\sin^2 f}{2r^2} - m^2(1 - \cos f)\right) \tag{2.27}$$

The energy (2.2) is given by:

$$E[f] = 2\pi g \int r dr \left(\frac{1}{2}\dot{f}^2 + \frac{1}{2}f'^2 + \frac{\sin^2 f}{2r^2} + m^2(1 - \cos f)\right) \tag{2.28}$$

The topological charge (2.4) is given by:

$$n[f] = -\frac{1}{2} \cos f|_{r=0}^{\infty} \tag{2.29}$$

We fix the boundary condition for f as $f(r = 0, t) = \pi$ and $f \to 0$ as $r \to \infty$, so we have unit topological charge and vacuum at large distance. Vanishing variation of (2.27) with respect to f results in the equation of motion:

$$-\ddot{f} + f'' + \frac{f'}{r} - \frac{\sin 2f}{2r^2} - m^2 \sin f = 0 \tag{2.30}$$

[1] for further details see appendix C.

We get the same equation if we put the ansatz (2.26) into full set of equations of motion (2.16).[1] For $m^2 = 0$, the equation has the solution

$$f(r,t) = 2\arctan\frac{\lambda}{r} \tag{2.31}$$

Let us now consider the static case of equation (2.30), i.e. $\dot{f} = \ddot{f} = 0$. For small r, we can develop f and calculate the coefficients order by order from the equation of motion:

$$f(r) = \pi + a_1 r - \frac{a_1}{12}(2a_1^2 + 3m^2)\, r^3 + \ldots \tag{2.32}$$

a_1 is the only free parameter and arbitrary higher order terms can be obtained in that way.

For $r \to \infty$ there are 2 possible asymptotic values : $f \to 0, \pi$. Equation (2.30) at large distance is

$$f'' + \frac{f'}{r} - \frac{f}{r^2} \mp m^2 f = 0 \tag{2.33}$$

where the sign in front of m^2 depends on the asymptotic value of f. In the case $f \to 0$, the asymptotic solutions are modified Bessel functions. Only the exponentially decaying one is physically acceptable:

$$f \to K_1(mr) \simeq \frac{1}{\sqrt{mr}}\, e^{-mr} \tag{2.34}$$

In the case $f \to \pi$ the asymptotic solutions are ordinary Bessel functions:

$$f \to \alpha\, J_1(mr) + \beta\, Y_1(mr) \simeq \frac{1}{\sqrt{mr}} \sin(mr + \phi) \tag{2.35}$$

The oscillatory behavior of the last solution is physically unacceptable because it leads to infinite energy (2.28).

We numerically integrated equation (2.30) for a static field $f(r)$. As asymptotics we used the two solutions found above. The result is shown in Figure 2.2. We see that the solution starting from π at $r = 0$ tends to an oscillatory solution around π at large distance. On the other hand, the solution starting from an exponential at large r diverges at the origin. This result follows necessarily from Derrick's theorem: no static finite energy solutions can exist in this model with $m^2 \neq 0$.

[1] If this is the case, the ansatz is said to "go through" the equation of motion. It means that an extremum of $S[f]$ is also an extremum of $S[\phi]$. We can expect that our ansatz goes through because it respects symmetries of the original theory, such as rotation, iso-rotation and time translation.

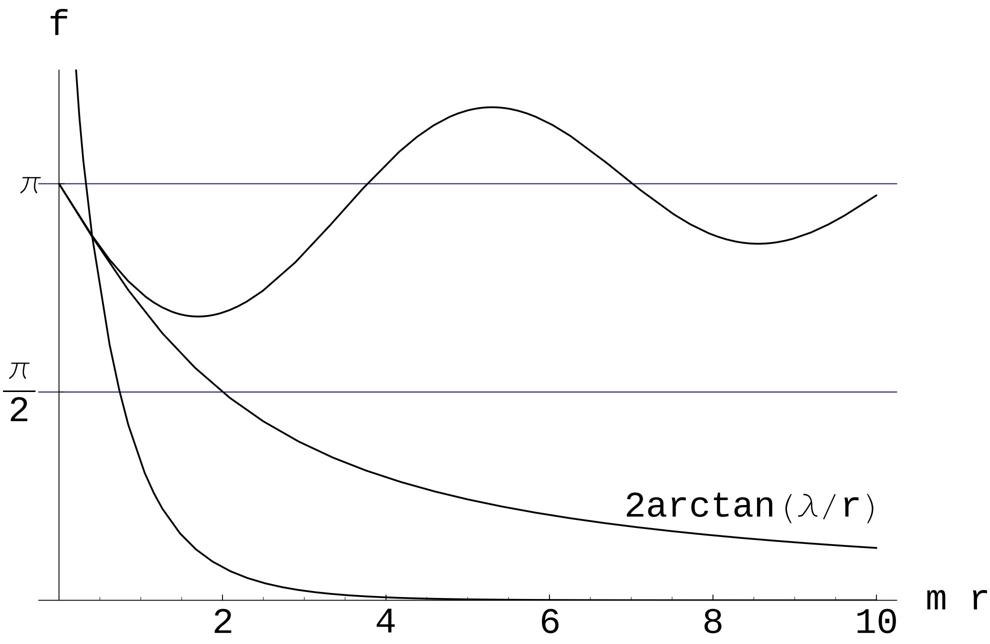

Figure 2.2: Static solutions to equation (2.30). No field starting from π at $r = 0$ and going to 0 as $r \to \infty$ can be found. This result is necessary from Derrick's theorem. As a reference, the static solution in the massless case is also plotted.

2.5.2 Lattice simulation

We addressed the question of decay in a lattice simulation of the partial differential equation (2.30). The result is that the soliton shrinks in finite time to a configuration which has all the energy concentrated near the origin. The numerical integration scheme breaks down at this point. However, it seems save to conclude that the final configuration is indeed singular.

It is convenient to express length and time in units of m^{-1} in this section. The dimensionless variables $x = m\,r$ and $\tau = m\,t$ will be used.

Definition of the soliton size In the general case of a time dependent solution, the "size" of the soliton has to be defined. There is more than one possibility. One definition is the mean radius over the energy density:

$$\langle x \rangle = \frac{2}{\pi} \frac{\int_0^\infty xdx\, x\, T^{00}(x,\tau)}{\int_0^\infty xdx\, T^{00}(x,\tau)} \tag{2.36}$$

where $T^{00}(x,\tau)$ is the energy density per surface at radius x. It is amusing to note that the time derivative of this size is given by the total field momentum: $\frac{d\langle x \rangle}{d\tau} = \frac{2}{\pi E} \int_0^\infty xdx\; T^{0x} = \frac{2}{\pi E} \int_0^\infty xdx\; \dot{f} f'$.

Another definition is taken from the Skyrme model. In this model, the the topological number density is interpreted as the baryon density. The mean square radius of the baryon is defined as the mean r^2 with respect to the baryon density. In our model, this corresponds to:

$$\Lambda^2 = \frac{1}{2}\int_0^\infty x dx \, (1 - \cos f(x,\tau)) \tag{2.37}$$

Note that this "size" is exactly the symmetry breaking term of our model. This definition of the size will be particularly useful in the quantum theory.

Since the total energy is conserved and we are considering a unit charge configuration, the maximum energy density at the center may also be used to define the spatial extension of the soliton:

$$l(t) = \sqrt{\frac{4\pi g}{T^{00}(x=0,t)}} \tag{2.38}$$

If we take for example the static solution $f(x,t) = 2\arctan\frac{\lambda}{x}$, the above definitions yield $\langle x\rangle = l = \lambda$. Λ^2 diverges logarithmically because the static solution has a x^{-1} tail. Definition (2.37) only makes sense in the massive case when the solution has an exponentially decaying tail.

Initial configuration The above idea of placing a static soliton and then raising the mass has the problem that we effectively add infinite energy to the system: the integral proportional to m^2 in (2.28) diverges for the static solution. We have to adopt a different approach.

At large x, the field should go to the static asymptotic solution (2.34). At small distance, the static field of the massless model is a good approximate solution for the massive case. This is seen for example in figure 2.2. We connect the two solutions at some point $x = X$ by the requirement of continuity up to the second derivative :

$$f(x,\tau=0) = \begin{cases} f_0(x), & x < X \\ f_1(x), & x > X \end{cases} \tag{2.39}$$

with

$$f_0(x) = 2\arctan\tfrac{\lambda}{x} \tag{2.40a}$$

$$f_1(x) = \tfrac{\alpha}{\sqrt{x-x_0}}\, e^{-(x-x0)} \tag{2.40b}$$

The connection conditions are:

$$f_0(X) = f_1(X) \tag{2.41a}$$

$$f_0'(X) = f_1'(X) \tag{2.41b}$$

$$f_0''(X) = f_1''(X) \tag{2.41c}$$

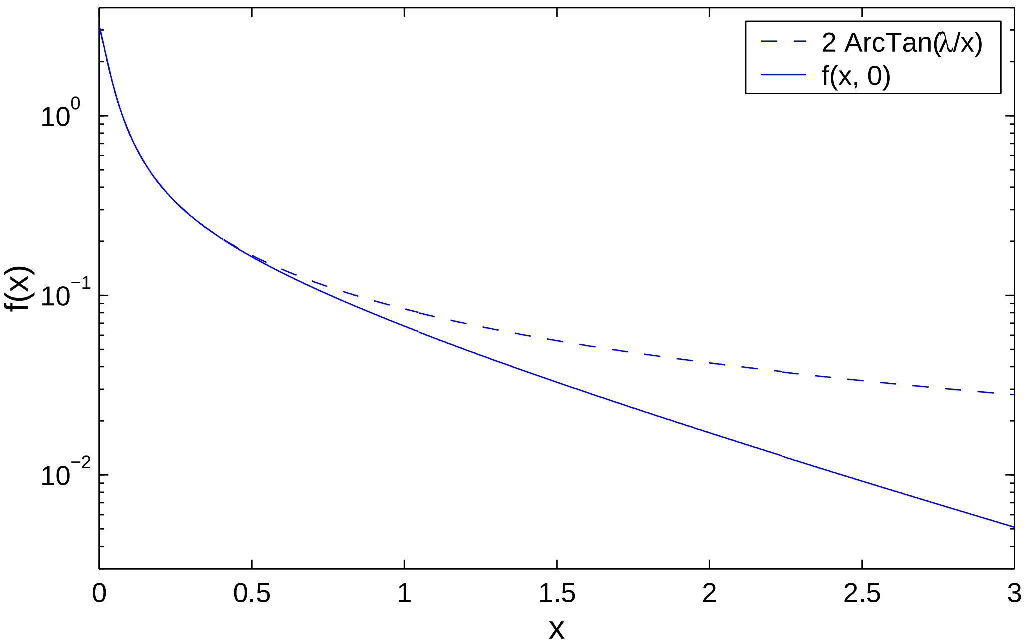

Figure 2.3: An initial configuration for the lattice simulation. For $x > X \simeq 0.3$, an exponential tail is taken. the boundary conditions are fixed at $x = 0$ and $x = 5.15$. The size of this configuration is $\lambda = \sqrt{7 \cdot 10^{-5}} \simeq 10^{-2}$.

We have 3 parameters α, x_0 and X and 3 equations (2.41). In the case $\lambda \ll 1$ we can develop the solution in λ. For example, $X \simeq 1 - \frac{2}{\sqrt{2}} + O(\lambda)^2$. The parameters are obtained numerically (Newton-Raphson) to arbitrary accuracy as a function of the initial size λ. (We will call the parameter λ in (2.40a) the size of the initial configuration)

It is clear that any (topologically nontrivial) initial configuration could be taken in principle. We prefer, however, to take the above initial configuration because it is close to static.

Integration algorithm A staggered leapfrog algorithm is used to integrate equation (2.30). The accuracy is second order both in time and space. The mass $m^2 = 7 \cdot 10^{-5}$ is taken for the simulation. Time steps are $dt = 10^{-3}$. Space steps are $dr = 2 \cdot 10^{-3}$ at the origin and doubled twice at $r = 25$ and $r = 136$. This is justified by the fact that the field and its derivatives are very small at large r. Fixed boundary conditions are applied at $r = 0$ and $r_{max} = 616$, the total number of lattice points in space is thus 10^5. We have $r_{max} \simeq 5\, m^{-1}$; a typical run has $t_{max} \simeq 1.8\, m^{-1}$. The waves travel at the speed $\frac{dx}{d\tau} < 1$. As a consequence, the effects from the fixed boundary condition at r_{max} are not felt by the soliton (size $\ll m^{-1}$ in our calculations).

Results of lattice simulation The field configuration at different times is plotted in figure 2.4. The derivative near the center gets bigger as f approaches zero rapidly at the outside. The evolution of the energy density in figure 2.5 shows a more and more pronounced concentration near the origin. Figure 2.6 displays the different parts of the total energy as a function of time. These quantities have a discontinuity at the time $\tau \simeq 1.6$. The total energy is not conserved at this point. This shows that the numerical integration breaks down; the field gradient gets too big.

The evolution of the size as defined above is plotted in figure 2.7. The size goes practically to zero at $\tau \simeq 1.6$. The different definitions for the size evolve proportionally to each other. In figure 2.8, the evolution of the size is shown for different initial configurations, labelled by λ. In this figure, the sizes are normalized to 1 at $\tau = 0$. The corresponding normalized velocities of shrinking are plotted in figure 2.9. The shrinking is uniformly accelerated up to $\tau \lesssim 0.4$ at a rate of $\partial_\tau^2 \langle x \rangle \simeq 8 \cdot 10^{-3} \lambda$. The initial acceleration is quite accurately proportional to the initial size. The velocity of collapse stabilizes at $\tau \simeq 1.1$ at a rate of $\partial_\tau \langle x \rangle \simeq 6 \cdot 10^{-3} \lambda$. The total time to collapse $\Delta\tau \simeq 1.6$ depends only little on the initial size of the soliton, at least for initial sizes in the range considered: $1 \leqslant \lambda \leqslant 30$.

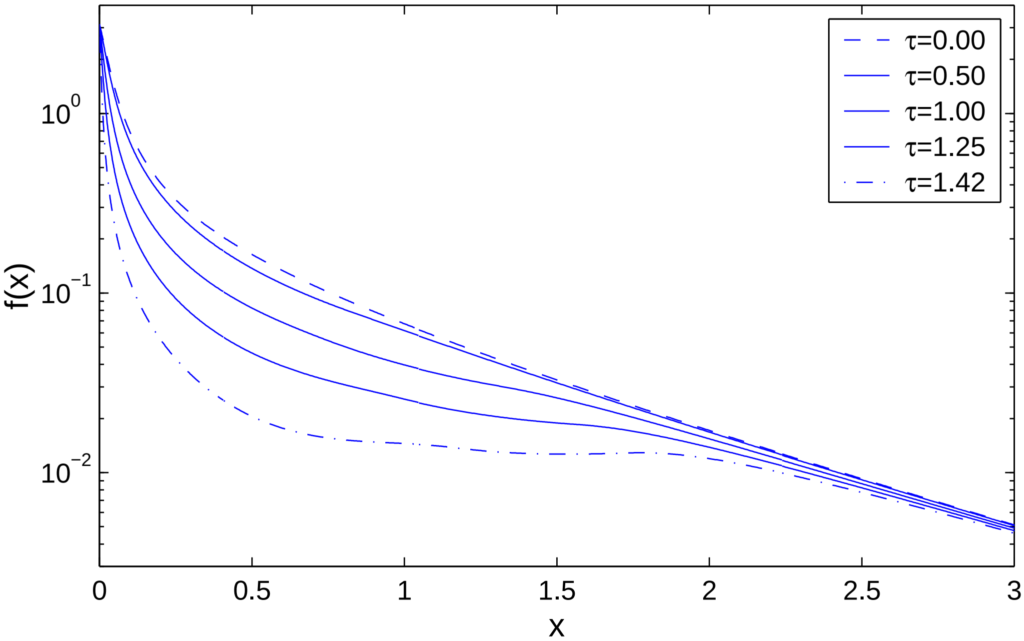

Figure 2.4: Evolution of a field with initial size $\lambda = 5\sqrt{7 \cdot 10^{-5}} \simeq 5 \cdot 10^{-2}$.

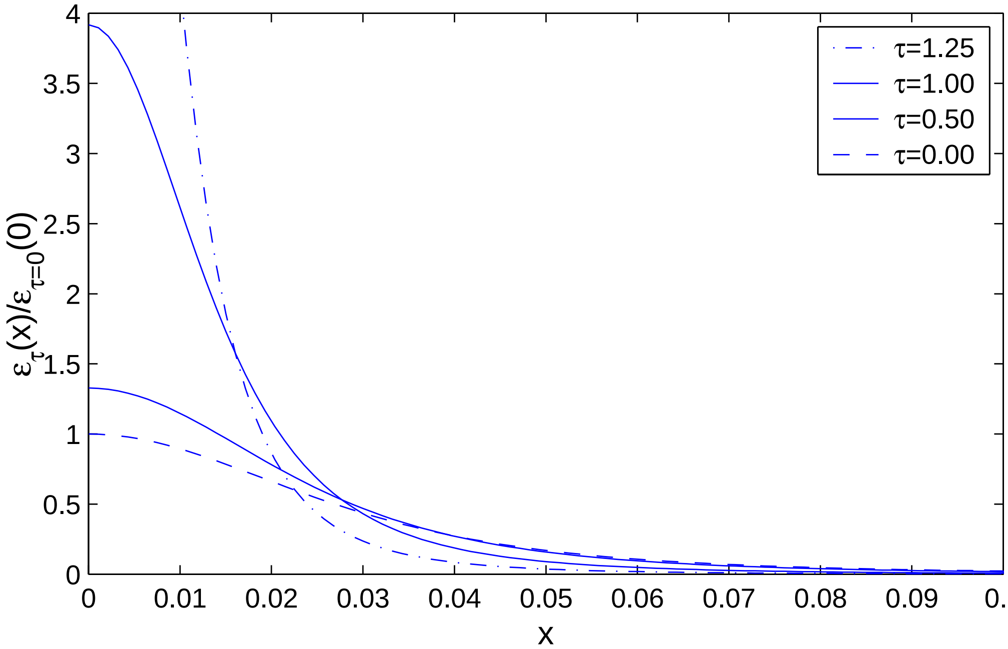

Figure 2.5: Evolution of the energy density of a field with initial size $\lambda = 5\sqrt{7 \cdot 10^{-5}}$. The densities are normalized to the initial density at the origin.

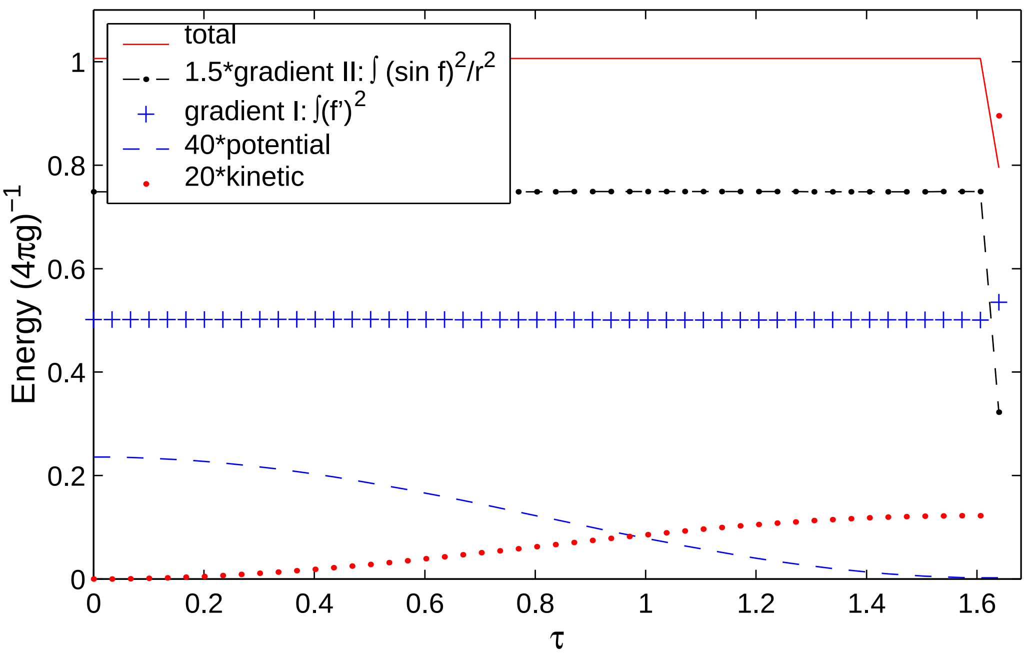

Figure 2.6: Evolution of the energy integrals. A discontinuity is encountered at time $\tau \simeq 1.6$ when the numerical integration breaks down.

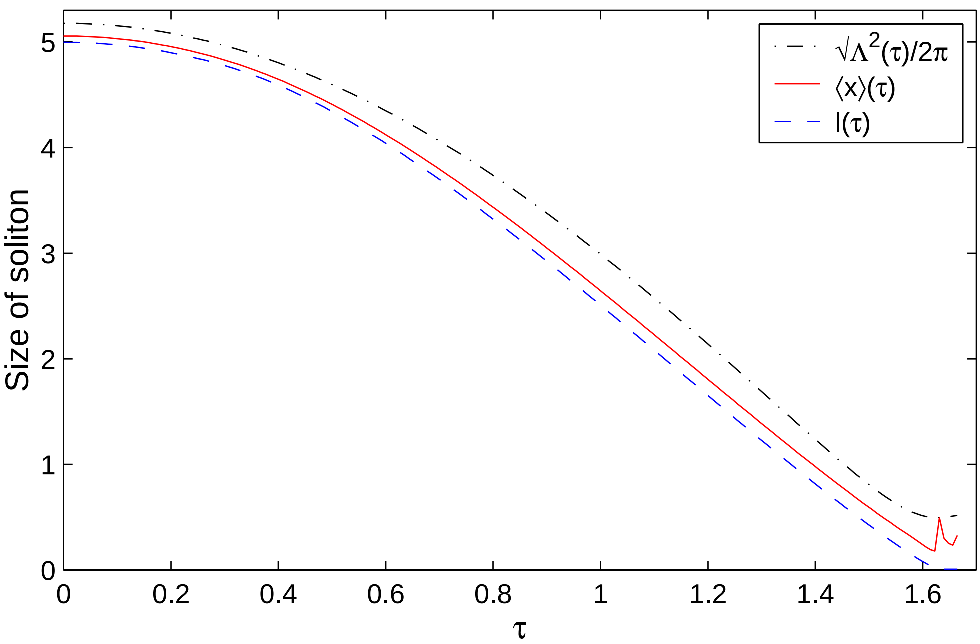

Figure 2.7: Evolution of the soliton size. The initial size is $\lambda = 5\sqrt{7 \cdot 10^{-5}}$. The different definitions of the size give very similar results.

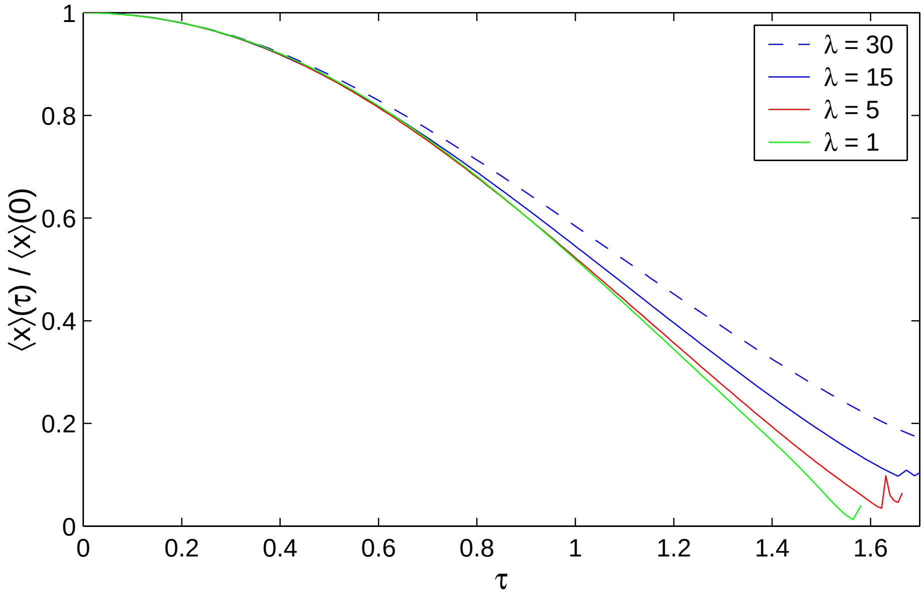

Figure 2.8: Evolution of size for different initial sizes λ (units $\sqrt{7 \cdot 10^{-5}}$). The sizes shown are normalized to 1 at $\tau = 0$.

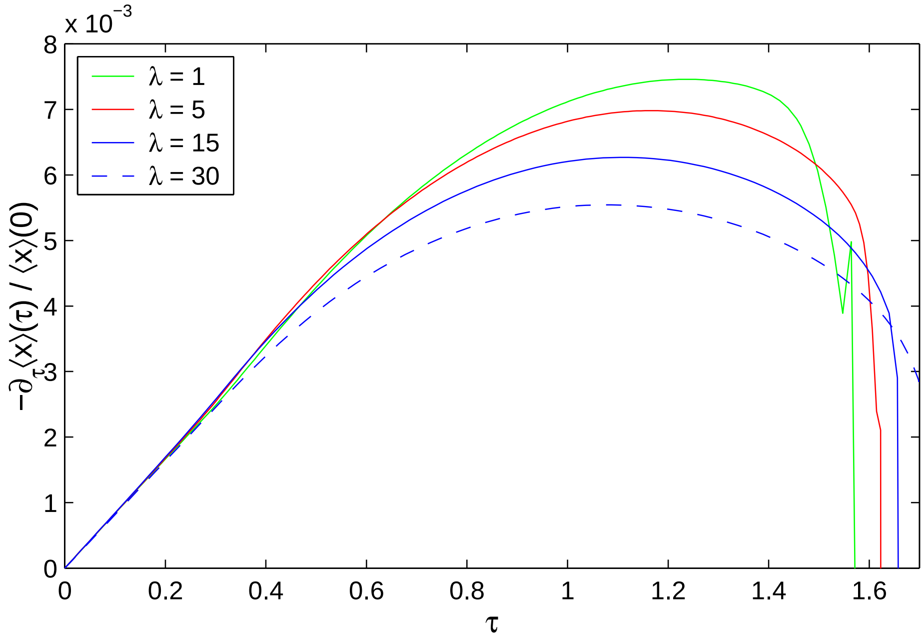

Figure 2.9: Evolution of normalized velocities of collapse for different initial sizes λ (units $\sqrt{7 \cdot 10^{-5}}$). The collapse is uniformly accelerated up to $\tau \simeq 0.4$. After that, the soliton shrinks with approximately constant velocity, $\partial_\tau \langle x \rangle \simeq 6 \cdot 10^{-3} \lambda$. The total time to collapse $\Delta\tau \simeq 1.6$ depends little on the initial size of the configuration.

3 Quantum Theory

In the last section we discussed the classical field theory of the model. We concluded that a topologically nontrivial initial configuration collapses to a singular configuration in a finite time interval. The classical evolution goes through a state which has all its energy concentrated at the origin. It is clear that such a state cannot exist quantum mechanically. In quantum mechanics, the fields and their momenta are non-commuting operators and the relation $\Delta p \Delta q \gtrsim \hbar$ has to hold necessarily (Δp and Δq are the uncertainties in momentum and location respectively). The construction of a quantum theory is thus crucial for the understanding of the model. There is, however, the delicate point of renormalisation which has to be discussed first.

This section is organized as follows: after a remark on the renormalizability of our model, we briefly discuss results of one-loop calculations which have been obtained by several authors. In the main part of the section, we present our results on the quantum mechanical stabilization of the collapsing soliton. The method we apply has been proposed for the Skyrme model in 4 dimensions.

3.1 Renormalizability

The first point concerns *renormalizability.* As is familiar from calculations in quantum theories with an infinite number of degrees of freedom, physical quantities often diverge. These infinities are removed by adding appropriate counterterms to the lagrangian density. This is however not possible in our 3-dimensional model. In order to illustrate this, let us discuss the structure of the vacuum sector of the model.

The model in terms of the fields $f(\mathbf{x}, t)$ and $h(\mathbf{x}, t)$ is given in (2.11). We rescale the fields with $\alpha = g^{-1/2}$:

$$f \to \alpha f \tag{3.1a}$$

$$h \to \alpha\, h \tag{3.1b}$$

With this redefinition, the lagrangian is

$$L = \int d^2x \; \frac{1}{2}(\partial_\mu f)^2 + \frac{1}{2}(\sin \alpha f)^2 (\partial_\mu h)^2 - \left(\frac{m}{\alpha}\right)^2 (1 - \cos \alpha f) \tag{3.2}$$

We now expand in the fields in order to apply the usual Feynman graph technique of perturbation theory. The classical equation of motion for the field f, (2.16a), decouples from h to lowest order and f is simply a particle with mass m. The field h is the massless Goldstone boson. By expanding (3.2) in the fields, we find the following vertices:

$$g^{-n}\,(\partial_\mu h)^2 f^{2n} \tag{3.3a}$$
$$m^2\,g^{-n+1}\,f^{2n} \tag{3.3b}$$

with n a positive integer. $2n$ lines f and 2 lines h attach to a vertex (3.3a). $2n$ lines f attache to a vertex (3.3b). These interactions are clearly nonrenormalizable: The vertex parameter has negative energy dimension. In order to get finite results out of perturbation theory, we would have to include an infinite number of counterterms in the lagrangian. However, we consider the model to be an effective theory, valid only at low energies (as it is done in the Skyrme model). The region of validity becomes clear by looking at the cross section in the tree approximation. From dimensional arguments the cross section corresponding to the vertices (3.3) are proportional to

$$\sigma \propto \frac{1}{E_{CM}}\,\Big(\frac{E_{CM}}{g}\Big)^n\,\frac{\mathbf{P}_{h_1}\cdot\mathbf{P}_{h_2}}{E_{CM}^2} \tag{3.4a}$$
$$\sigma \propto \frac{1}{E_{CM}}\,\Big(\frac{m}{E_{CM}}\Big)^2\,\Big(\frac{E_{CM}}{g}\Big)^{n-1} \tag{3.4b}$$

where $\mathbf{P}_{h_1}$ and $\mathbf{P}_{h_1}$ are the momenta carried by the goldstone particles; E_{CM} is the center of mass energy of the particles. Equation (3.4a) shows that the parameter g sets a natural energy cutoff: if we consider only particles at energies $E_{CM} \ll g$, then we are allowed to neglect high n vertices. Furthermore, from (3.4b) we learn that $m \ll E_{CM} \ll g$ in order to have a consistent theory. This provides us with natural infrared m and ultraviolet g cutoffs which regularize the model.[1]

3.2 Vacuum Energy

The calculation of vacuum fluctuations in presence of a background field is important in soliton theory. The soliton is considered as a classical background and one calculates semiclassical corrections to quantities like the energy. This semi-classical approach is equivalent to sum over 1-loop vacuum Feynman diagrams in perturbation theory. This is a difficult task in general which has to be addressed numerically in most cases. The general techniques are explained by Weigel et al. [20, 21]. In the following I will briefly introduce the principal idea and state the known results for the static hedgehog soliton of our model.

[1] Equivalently, it sets a spatial short range lattice spacing to g^{-1}.

It is convenient here to use imaginary time *it* and the euclidian metric $g_{\mu\nu} \rightarrow -\delta_{\mu\nu}$. The generating functional of a model in quantum field theory can be written as a Feynman path integral over the fields:

$$Z[0] = \int D\phi \, e^{-S[\phi]} \tag{3.5}$$

Suppose we have a classical field ϕ_c, solution of the equation of motion $\frac{\delta S[\phi]}{\delta\phi}|_{\phi=\phi_c} = 0$. The semiclassical approximation means that we set $\phi \simeq \phi_c + \eta$. and expand for small η. The integration can now be shifted[1]to η and the gaussian integration performed:[2]

$$\begin{aligned} Z[0] &\simeq \int D\phi \, e^{-S[\phi_c+\eta]} \\ &\simeq \int D\eta \, \exp(-S[\phi_c] - \frac{1}{2}\int dx\, dy\, \eta(x)\frac{\delta^2 S[\phi_c]}{\delta\phi(x)\delta\phi(y)}\eta(y)\ ) \\ &= N(\det S''[\phi_c])^{-1/2}\exp(-S[\phi_c]) \\ &= N\exp(-S[\phi_c] - \frac{1}{2}\log\det S''[\phi_c])\ ) \end{aligned} \tag{3.6}$$

here we have used the notation $\frac{\delta^2 S[\phi]}{\delta\phi_2}|_{\phi=\phi_c} = S''[\phi_c]$. In order to get rid of unphysical infinities (e.g. integration over infinite volume) one should divide the generating functional by its value on the vacuum field, noted 0. Hence, the *effective action* including quantum effects to 1-loop order can be written as

$$\begin{aligned} S_{eff}[\phi] &= S[\phi] - S[0] + \frac{1}{2}\log\frac{\det S''[\phi]}{\det S''[0]} \\ &= S[\phi] + \frac{1}{2}\mathrm{Tr}(\log S''[\phi] - \log S''[0]) \end{aligned} \tag{3.7}$$

where $S[0]$ has been taken to be 0. The determinant and trace are understood as the product respectively the sum over all eigenvalues of the operator $S''[\phi]$, evaluated at the classical field ϕ.[3] For a model with the (euclidian) lagrangian density $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + V(\phi)$ this operator is $S''[\phi] = -\partial_\mu\partial^\mu + V''(\phi)$.

[1]The Jacobian resulting from this shift is in general not one. Fortunately, this multiplicative constant is not important in most cases because we normalize with respect to the vacuum configuration $\phi_c = 0$.

[2]We consider only bosonic fields here. For fermions, signs would change.

[3]Zero eigenvalues are to be excluded from the determinant and treated separately. We will not discuss the issue of zero-modes here. Details can be found in [1].

In the case of a time independent background field, the operator S'' can be separately diagonalized in space and time:

$$(\omega_k^2 + \partial_i^2 - V''(\phi))\, \eta_k(x) = 0 \tag{3.8a}$$
$$\log\det S'' = \mathrm{Tr}\log(\partial_t^2 + \omega_k^2) \tag{3.8b}$$

(3.8b) can now be calculated: the eigenfunctions are $\exp(\nu t)$ with eigenvalues $\nu^2 + \omega_k^2$ and the trace can be written as an integral:

$$\log\det S'' = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \log(\nu^2 + \omega_i^2) = \sum_i \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \log(1 + (\omega_i/\nu)^2) + 2\log\nu \tag{3.9}$$

The last term is divergent and can be absorbed in the factor N. The remaining integral is readily performed. The final result is

$$S_{eff} = S[\phi] + \frac{1}{2}\sum_i (\omega_i - \omega_i^0) \tag{3.10}$$

If we go back to real time, we see that the loop contribution to the action (3.7) contributes to the energy of the static soliton ϕ_c as:

$$E_{eff} = E[\phi_c] + \frac{1}{2}\sum_i (\omega_i - \omega_i^0) \tag{3.11}$$

Hence the name vacuum or casimir energy.

As a result, the effective energy to 1-loop order in presence of a static background field is determined by the sum over the eigenvalues of the Schrödinger equation (3.8a). This sum is potentially divergent and has to be regularized. Let us just note that the determinant (3.8b) can also be linked to the phase shift of the scattering waves of (3.8a) by Levison's theorem. In this way, the sum over eigenvalues can be circumvented. More details can be found in [3, 20, 21].

We have already written the Schrödinger equation (3.8a) for our static soliton in the $m^2 = 0$ case when we discussed stability. In fact, the soliton is classically stable if this Schrödinger equation has no bound states.

The question of vacuum corrections to the static soliton of our model has first been addressed by Rodriguez [22]. More recent and thorough studies which include renormalization to loop order have been worked out by Moss [23] and Walliser et al. [24]. The result of these authors is that the static soliton (2.31) has the energy to loop order

$$E = 4\pi g - \frac{\alpha}{\lambda} \tag{3.12}$$

The numerical constant is $\alpha \simeq 0.248$ (Moss) and $\alpha \simeq 0.5$ (Walliser). As a result, small hedgehog solitons have less energy and the vacuum fluctuations lead to a shrinking of the soliton. However, these authors do not discuss what happens when the soliton gets very small.

3.3 Quantum stabilization

Mechanisms of quantum stabilization have been proposed in the context of the 4 dimensional Skyrme model by Chepilko et al. [25, 26, 27], Preston et al. [28, 29], Balakrishna [30] and Carlson [31, 32]. We will use here the ideas of Chepilko and colleagues.

We saw that there are static solutions of the form $f(r,t) = F(\frac{r}{\lambda})$ with $F(x) = 2\arctan(x^{-1})$ in the massless case. In the massive case we have to look for a time dependent solution. However, the simple ansatz of a time dependent size $\lambda = \lambda(t)$ fails, because it has infinite energy. The idea of Chepilko is to suppose that the profile $F(x)$ differs from the static case, but the soliton still changes only its scale during evolution

$$f(r,t) = F\big(\frac{r}{\lambda(t)}\big) \tag{3.13}$$

In order to have a configuration with nontrivial topology, the profile F must satisfy the boundary conditions:

$$F(x) = \begin{cases} \pi & x = 0 \\ 0 & x \to \infty \end{cases} \tag{3.14}$$

The classical equation of motion (2.18) can certainly not be used to determine the profile F. However, we can put the ansatz (3.13) into the action of our model (2.27) and integrate over space. The result is an effective action for the dynamical variable $\lambda(t)$. The parameters of this action will depend on the profile. A quantum theory can now be constructed for λ. Physically admissible wave functions $\psi(\lambda)$ should not only be square integrable but also vanish at $\lambda = 0$. The reason is that $\lambda = 0$ corresponds to a singular field configuration which should have zero probability.

$$\psi(0) = 0 \tag{3.15a}$$

$$\int_0^\infty d\lambda\, |\psi(\lambda)|^2 < \infty \tag{3.15b}$$

Chepilko et al. argue that the profile F should be obtained from the equation

$$\langle n|\frac{\delta L}{\delta F}|n\rangle = 0 \tag{3.16}$$

Where L is the lagrangian (operator) and $|n\rangle$ are energy eigenstates. Hence, for every quantum state n we have to find a profile which satisfies this equation. Chepilko's derivation of the profile equation (3.16) is not entirely clear, at least to the author. However, it is shown in appendix D that, for a large class of models including the one discussed here, equation (3.16) is equivalent to

$$\frac{\delta E_n}{\delta F} = 0 \tag{3.17}$$

where E_n is the energy of the eigenstate n. This equation is at least reasonable. It requires the energy to be stationary under variations of the profile. A quantum stabilized soliton is in a stationary state and the classical parameters should adjust to minimize the energy.

Let us turn to our model. The lagrangian corresponding to (2.27) and using the ansatz (3.13) is given by:

$$L/(2\pi) = \frac{g}{2}\, A[F]\, \dot{\lambda}^2 - g\, B[F] - \frac{m^2 g}{2}\, C[F]\, \lambda^2 \tag{3.18}$$

where the coefficients are given by

$$\begin{aligned} A[F] &= \int_0^\infty x^3 dx\; F'(x)^2 && \text{(3.19a)} \\ B[F] &= \frac{1}{2}\int_0^\infty x\, dx\; (F'(x)^2 + \frac{\sin^2 F(x)}{x^2}) && \text{(3.19b)} \\ C[F] &= 2\int_0^\infty x\, dx\; (1 - \cos F(x)) && \text{(3.19c)} \end{aligned}$$

(3.18) is a harmonic oscillator. The corresponding hamiltonian is

$$H/2\pi = \frac{P_\lambda^2}{2A[F]g} + g\, B[F] + \frac{m^2 g}{2}\, C[F]\, \lambda^2 \tag{3.20}$$

with the canonical momentum $P_\lambda = A\, g\, \dot{\lambda}$. The oscillator has the mass $M = A\, g$ and the frequency $\Omega = m\sqrt{\frac{C}{A}}$. The well known spectrum is

$$E_n = 2\pi(g\, B + m\sqrt{\frac{C}{A}}\; (n + \frac{1}{2})\,) \tag{3.21}$$

The condition of vanishing wave function at $\lambda = 0$ restricts n to odd integers.

3.3.1 Profile equation

The above discussion supposes the existence of a function F, satisfying the profile equation. Let us first remark an interesting property of the limit $m^2 \to 0$. In this limit, the spectrum (3.21) reduces to the classical energy $2\pi g\, B$. The profile equation $\delta_F B = 0$ is nothing else than the static equation of motion (2.30) with the known solution (2.31). We recover the classical case in the massless limit.

We want now explore the full profile equation (3.16). It is

$$\frac{1}{2}\frac{m}{g}\sqrt{\frac{C}{A^3}}(n+\frac{1}{2})\,\delta_F A - \delta_F B - \frac{m}{g}\frac{1}{2\sqrt{AC}}(n+\frac{1}{2})\,\delta_F C = 0 \tag{3.22}$$

To analyze this equation, we first change variable:

$$z = \left(\frac{m}{g}\sqrt{\frac{C}{A^3}}\,(n+1/2)\right)^{1/2} x \tag{3.23a}$$

$$\tilde{F}(z) = F(x) \tag{3.23b}$$

The profile equation (3.22) gets

$$(z(z^2-1)\tilde{F}'(z))' + \frac{\sin 2\tilde{F}(z)}{2z} + \beta[\tilde{F}]\sin\tilde{F}(z)z = 0 \tag{3.24}$$

where $\beta \geq 0$ is given by

$$\beta[\tilde{F}] = \frac{A}{C} \tag{3.25}$$

We have used here the fact that $\beta[F] = \beta[\tilde{F}]$. Notice also that $B[F] = B[\tilde{F}]$ (The classical energy is scale invariant).

Let us first write the asymptotic solutions to (3.24). For large z and taking into account $\tilde{F} \to 0$, these solutions are

$$\tilde{F} \to \begin{cases} c_1\, z^{-2} + c_2 & \beta = 0 \\ c_1\, z^{-1-\sqrt{1-\beta}} + c_2\, z^{-1+\sqrt{1-\beta}} & 0 < \beta \leq 1 \\ c_1\, z^{-1}\sin(z\,\sqrt{\beta-1}) + c_2\, z^{-1}\cos(z\,\sqrt{\beta-1}) & \beta > 1 \end{cases} \tag{3.26}$$

where c_1 and c_2 are integration constants. Finiteness of the energy (3.21) restricts us to the case $0 \leq \beta \leq 1$.

Equation (3.24) is a fairly complicated integral-differential equation. We have numerically integrated equation (3.24) for fixed values of β with appropriate boundary conditions. The numerical procedure is explained in appendix E. We find smooth solutions for every $0 \leq \beta \leq 1$. Unfortunately,

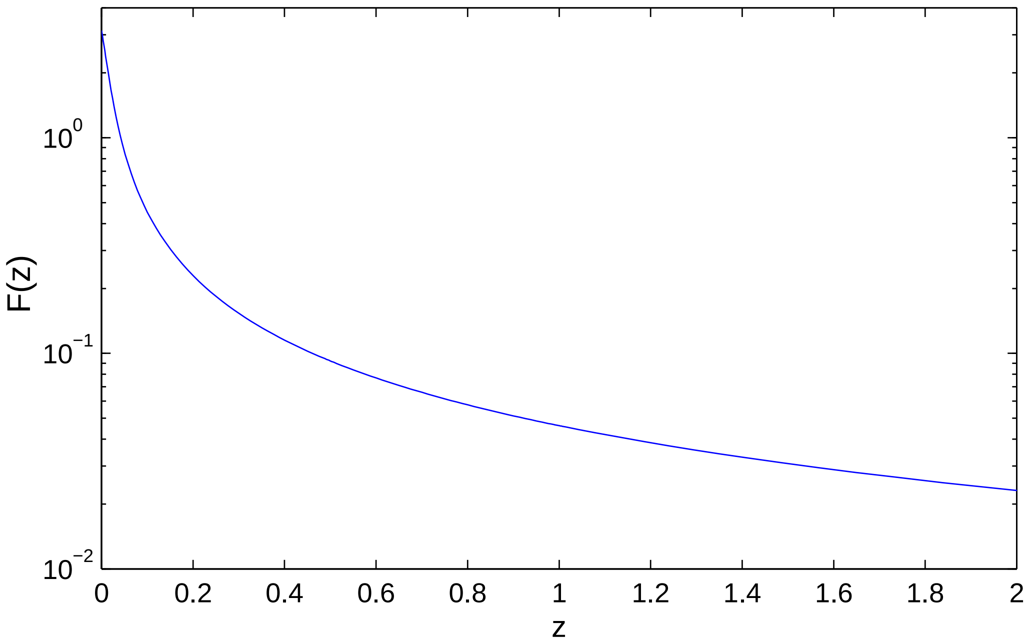

Figure 3.1: A solution to the profile equation (3.24) with $\beta = 1$. The differential equation has a singular point at $z = 1$. However, smooth solutions can be found.

all found solutions have $c_2 \simeq 10^{-2}$, i.e. F does not vanish fast enough at infinity to make A and C finite. However, The ratio A/C is finite and can be calculated explicitly in this case (appendix E):

$$\frac{A}{C} = (-1 + \sqrt{\beta - 1})^2 \tag{3.27}$$

In order to have a consistent solution, we have also to satisfy the relation (3.25). This gives us two solutions: $\beta = 0$ and $\beta = 1$. The first one has infinite energy so we are left with $\beta = 1$. The profile for $\beta = 1$ is plotted in figure 3.1. The corresponding classical energy is $B[F] \simeq 2.0011$.

We have now a solution $\tilde{F}$ to the profile equation, but $A[\tilde{F}]$ and $C[\tilde{F}]$ diverge logarithically. One can easily verify that this implies that also $A[F]$ and $C[F]$ diverge. Hence, we are left with the harmonic oscillator (3.20) with finite frequency $\Omega = m\sqrt{A/C}$ but infinite mass $M = g\,A$. A desperate situation at first sight. The particle λ, being infinitely heavy, is constraint to sit at the origin and the soliton has decayed. The quantity $\langle n|\lambda^2|n\rangle$, which could be taken as the soliton size, vanishes. On the other hand, a physical quantity like the energy does not care at all: it is finite and has different levels. $\langle\lambda^2\rangle$ is not the only possible definition of the size. It vanishes because the profile F has a long tail. A reasonable definition of the soliton size should rather be large in the limit of a long range profile.

A solution is presented by an idea from the Skyrme model. As we already discussed, the topological charge density in this model is interpreted as the particle density. If we adopt this point of view, we can calculate the matrix element of the mean square radius (2.37):

$$\begin{aligned} \langle n|\Lambda^2|n\rangle &= \int_0^\infty d\lambda\, |\psi_n(\lambda)|^2\, \frac{1}{2} \int_0^\infty rdr\, (1 - \cos F(\frac{r}{\lambda})) \\ &= \int_0^\infty d\lambda\, |\psi_n(\lambda)|^2\, \lambda^2\, \frac{1}{2} \int_0^\infty zdz\, (1 - \cos F(z)) \\ &= \int_0^\infty d\hat{\lambda}\, |\hat{\psi}_n(\hat{\lambda})|^2\, \frac{\hat{\lambda}^2}{mg\sqrt{AC}}\, \frac{1}{4}\, C[F] \end{aligned} \tag{3.28}$$

Finally, taking into account $A/C = \beta = 1$ and the properties of the harmonic oscillator, we get

$$\langle n|\Lambda^2|n\rangle = \frac{1}{4mg}\, (n + \frac{1}{2}) \tag{3.29}$$

Inspired by this result, we are led to the conjecture that all physically observable quantities of this model depend only on the ration A/C while A and C may diverge. One might argue that, in this case, the change of variable (3.23) is not permissible. This issue is resolved if we regularize the integrals A and C by introducing a large cutoff (typically proportional to m^{-1} because this is the infrared cutoff of our model). The Integrals are then finite and the change of variable can be done. Our final results do not depend on this cutoff, provided it is large.

Of course, these results would be useless if the quantum stabilized soliton turned out to be smaller then the lattice spacing g^{-1}. However, we have $\sqrt{\langle\Lambda^2\rangle}/g^{-1} \simeq \sqrt{g/m} \gg 1$.

3.3.2 Vacuum fluctuations in the massless case

Let us briefly discuss the case when vacuum fluctuations are taken into account. As mentioned previously, vacuum fluctuations around the static soliton in the massless model add a term $-\alpha/\lambda$ to the energy. This makes the classical soliton unstable. We can now apply exactly the same procedure as before and quantize the dilatation mode λ in the massless model. The resulting spectrum is that of the one-dimensional hydrogen atom [33]:

$$E_n = 2\pi g\, \big(B[F] - \frac{A[F]\alpha^2}{2n^2}\big) \tag{3.30}$$

However, if we write the profile equation for this model, we get the previous profile equation (3.24) with $\beta = 0$. As we discussed in the massive case, this leads to a divergent energy (B) and the soliton cannot exist. In reality, however, the vacuum contribution to the energy depends on the profile of the soliton. Supposing the λ^{-1}-law is still valid (which is reasonable from dimensionality) then the parameter α is a functional of F. This changes the profile equation. A conclusive answer in the massless case can only be given if the vacuum fluctuations are taken into account for a varying profile. We have not further explored this question.

3.4 Discussion

The results (3.21) and (3.29) show that the soliton is stabilized when quantum effects are taken into account. We obtained a quantum ground state and a tower of exited soliton states. The size is nonzero and the soliton is stabilized. If we restore cgs units, we get $\langle\Lambda^2\rangle \propto \hbar^2 c^2/(mg)$ where c is the speed of light. It is now apparent that the size of the soliton vanishes in the limit $\hbar \to 0$. This means that we have a quantum soliton which does not exist in the classical limit.

The physical meaning of this result is somewhat unusual: the quantum soliton is a superposition of solitons with different sizes. For example we can write the form factor of the quantum soliton in the energy state n as $f_n(r) = \int_0^\infty |\psi_n(\lambda)|^2 d\lambda\, F(r/\lambda)$. To our knowledge, such a type of soliton has only been studied very scarcely in the literature.

Let us briefly make a remark on the method. The ansatz (3.13) effectively reduces the infinite number of quantum degrees of freedom of the field theory to one quantum degree λ and a classical field F. This is certainly a very gross approximation and its validity is a priori debatable. Let us quote Carlson [31] who said in this context:

> Incorporating quantum effects one degree of freedom at a time is an inexact science. It is not a systematic expansion in powers of $\hbar$, but rather an expansion in what are viewed as the "important" degrees of freedom. Which degree of freedom are important, however, can only be decided *a posteriori*.

It is a central claim of this work that λ is an important, and even the most important quantum degree of freedom for small solitons. Any field configuration, independently of its actual form factor (its classical "shape") has to go through a singular configuration in order change its topology. In the example discussed in this work, the classical field goes through a configuration with

$f(r=0)=\pi$ and $f(r>0)=0$. The parameter λ in $f(r)=F(r/\lambda)$ seems to be a good measure of this singular behavior. For any smooth trial function F respecting the boundary conditions (3.14), we expect the classical field to go through the configuration with $\lambda=0$. The ansatz used is a very general one for such a field near the singularity.

Another important point is that the classical energy of the soliton does not depend on λ. The standard collective coordinate approach for quantizing zero-modes cannot be directly applied to the dilatation variable because of its non-normalizability. We solve this problem by introducing the profile function F. From then on, we proceed in accordance with the standard prescription for quantizing zero-modes (see e.g. [1]). It is a common practice in field theory to treat zero-modes separately from other quantum effects. And normally, their effect is more important than other degrees of freedom. It is clear that, in order to have a complete picture, we should also include the rotational zero-mode as a quantum degree of freedom. Our results on this mode are not included in this report because the corresponding profile equation has not been fully analyzed yet. However, its contribution to the energy are expected to be of the same order as the corrections due to the λ mode. There seems to be no indication that instability should enter in this way.

4 Conclusion and outlook

In the first section we examined the classical field equations. We concluded that the soliton shrinks to a configuration which has all its energy concentrated at the origin. This is an unphysical situation which is naturally prevented by quantum mechanics. We were left with two possibilities: either the soliton decays to radiation which would mean that a change in topology occurs. Or the shrinking soliton is stabilized at some stage by quantum effects. In order to construct a quantum theory, we reduced the field degrees of freedom. We picked out one quantum degree of freedom λ and the classical field F. With this reduction, we were able to show that a quantum stabilization indeed occurs.

It is clear that the present study leaves open questions. For example, more modes should be included into the quantum reduction and their effects should be compared. First of all, the rotational mode has to be considered. The next step would be to include the full vacuum corrections to the quantum soliton. As we discussed, this will most probably add a λ^{-1} potential into the reduced problem. Here also, the corresponding profile equation would have to be analyzed. Another point is the profile equation itself. Its role in the stabilization mechanism is not entirely clear. Furthermore, the relation to the profile equations proposed by other authors should be analyzed more closely.

Finally, let us remark that the quantum soliton could find interesting application in cosmology. The use of a quantum soliton in the brane world scenario has not been investigated yet. Our four dimensional world in the domain wall of a quantum soliton may have very unusual properties. We hope to explore this question in future work.

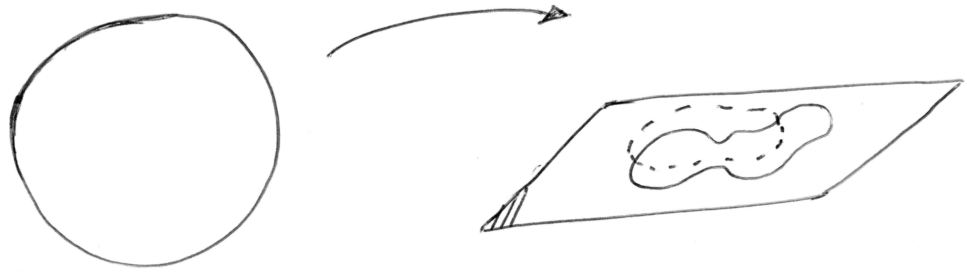

Figure A.1: Map of a circle to the plain. There is only one homotopy class because all circles can be deformed to one another.

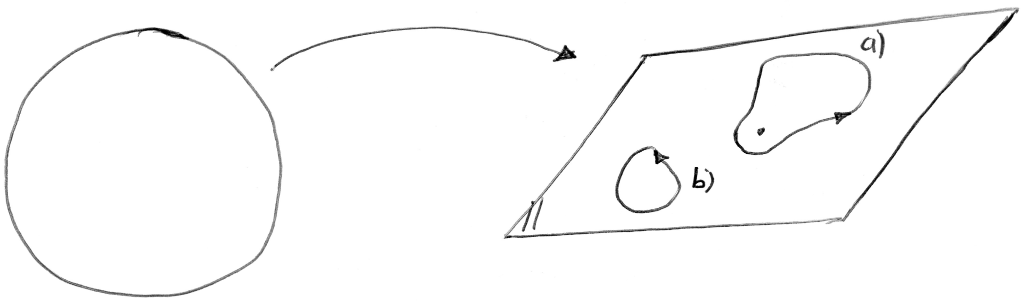

Figure A.2: Map of a circle to the plain with a hole. Homotopy classes are non-trivial.

A On winding numbers

The winding number is an index which allows us to classify the maps from one space into another. Consider for example the map which takes a circle S^1 to a plain $\mathbb{R}^2$ in figure A.1. We say that two maps belong to the same homotopy class if the image of one map can be continuously deformed to the image of the other. One can show that the set of homotopy classes has the structure of a group. In the example above we see easily that the circle on the plain can be deformed to any other circle. We say in this case that the homotopy group is trivial, i.e. it contains only one element. If we make the target space slightly more complicated, for example if we cancel one point, then the homotpy group will be non-trivial. In figure A.2 we see that the circle a) which goes around the point cannot be deformed to b) without cutting it into two pieces. Furthermore, the sense of rotation is now important: a circle which goes once around the point clockwise cannot be deformed to one turning anti-clockwise! The topological classification is as follows: we say that two maps are equivalent if the circle goes n times around the point turning clockwise. To any circle we can now associate an integer n: to circle a) $n = -1$ to circle b) $n = 0$. As far as notation is concerned, we call the homotopy group of maps from the circle to the plain $\pi_1(\mathbb{R} \setminus \{0\})$. The fact that it can be represented by an integer is written as $\pi_1(\mathbb{R} \setminus \{0\}) \equiv \mathbb{Z}$.

Let us now go on to the slightly more complicated topological space of this work. As is was discussed, fields $\boldsymbol{\phi}$ with finite energy have to go to a constant value at infinity, independent of the direction taken. This allows us to compactify the physical space $\mathbb{R}^2$ to the sphere S^2_{phys}. That is, $\mathbb{R}^2$ with the points at infinity identified is topologically equivalent to S^2_{phys}. Hence any finite energy configuration is a map

$$\boldsymbol{\phi} : S^2_{phys} \to S^2_{int} \tag{A.1}$$

The corresponding homotopy group is written $\pi_2(S^2_{int})$. Exactly analog to the example given above, to any configuration we can associate the number of times the physical space is "wrapped" around the internal space by the finite energy configuration.

In order to get an explicit expression of this number in terms of the field, consider the infinitesimal surface in physical space, $dx_1\, dx_2$. This surface will be mapped to $\|\partial_1\boldsymbol{\phi} \wedge \partial_2\boldsymbol{\phi}\|$ by $\boldsymbol{\phi}$. To get the winding number, we integrate this quantity over all physical space and divide it by the total volume of internal space, i.e. 4π:

$$n = \frac{1}{4\pi}\int_{S_2^{phys}} d^2x\; \boldsymbol{\phi}\cdot\partial_1\boldsymbol{\phi}\wedge\partial_2\boldsymbol{\phi} = \frac{1}{8\pi}\int_{S_2^{phys}} d^2x\; \epsilon^{abc}\epsilon_{ij}\phi_a\partial_i\phi_b\partial_j\phi_c \tag{A.2}$$

This is the expression used in the text.

B The $CP(1)$ formulation

There are different formulations of the $O(3)$ nonlinear σ-model than the one used in this work. The so called $CP(1)$ formulation is interesting for at least one reason: it allows one to obtain static solutions with arbitrary topological number (for $m^2 = 0$). Let us briefly discuss how this works.

The $O(3)$ σ-model is originally defined on a sphere S^2_{int}. We can project this sphere onto the complex plain $\mathbb{C}$. Taking the stereographic projection, this is the transformation:

$$w = w_1 + iw_2 = \frac{\phi_1 + i\,\phi_2}{1 - \phi_3} \tag{B.1}$$

Furthermore, one can associate the physical space $\mathbb{R}^2$ with the complex plain, $z = x_1 + i\,x_2$. We can now formulate a completely equivalent field theory in terms of the complex field $w(z,t)$, defined on the complex space.

Let us briefly return to the standard $O(3)$ formulation of the model. In section 2, we derived the inequality (2.7). This inequality allowed us to find the lower bound (2.8) on the energy functional. It is easy to see that, in the case $m^2 = 0$, this inequality (2.8) can only be saturated for a static field. We can try looking for such a field in a given topological sector. This field necessarily saturates the inequality (2.7), i.e.

$$\partial_i \phi_a \pm \epsilon^{abc} \epsilon_{ij} \phi_b \partial_j \phi_c = 0 \tag{B.2}$$

The sign in (B.2) will be identified with the sign of the winding number we want to obtain. A field satisfying (B.2) minimizes the energy in some topological sector and it is easily checked that it also satisfies the full set of field equations $\delta S/\delta\boldsymbol{\phi}$. However, it has to be noted that the inverse is not true in general. There might be higher local energy minima in a given topological sector. Note the formidable simplification of the problem: we have passed from the full equation of motion (second-order) to a system of first-order differential equations (B.2).

The equations (B.2) are particularly interesting in the $CP(1)$ language. They are:

$$\begin{aligned} \frac{\partial w_1}{\partial x_1} \mp \frac{\partial w_2}{\partial x_2} &= 0 \\ \frac{\partial w_1}{\partial x_2} \pm \frac{\partial w_2}{\partial x_1} &= 0 \end{aligned} \tag{B.3}$$

These are nothing else than the Cauchy-Riemann equations on the complex function w. The general solution of (B.3) is $w(x_1 + ix_2) = f(x_1 \pm ix_2)$, where

$f(z)$ is an arbitrary analytic function of z. In fact, singularities are allowed for the function $f(z)$, they simply mean that the field is on the south pole of S^2_{int}, i.e. $\phi_3 = -1$. However, a branch-cut is not allowed because the field would not be single-valued. Such functions with isolated poles but no essential singularities are *meromorphic functions*. We can restrict ourselves to functions w which are meromorphic in the variable $z = x_1 + ix_2$, i.e. to solutions with positive winding number n. The corresponding $-n$ solution can simply be obtained by inversion of one coordinate-axis.

It can be shown [12] that the most general minimum solution in a topological sector n is given by:

$$w(z) = \prod_i \Big(\frac{z - z_i}{\lambda}\Big)^{m_i} \prod_j \Big(\frac{\lambda}{z - z_j}\Big)^{n_j} \tag{B.4}$$

with $n = \sum m_i > \sum n_j$. We see that there are in general $2n$ arbitrary complex parameters z_i, z_j and λ. z_i define essentially the position and orientation of the different solitons. λ is the scale parameter which is due to the dilatation invariance of static solutions. In section 2 we discussed static hedgehog solutions with topological number n. In $CP(1)$ they are simply

$$w(z) = \left(\frac{\lambda}{z}\right)^n \tag{B.5}$$

C Derrick's theorem

Several field theories in 1+1 dimensions are known to possess static (classical) soliton solutions. The natural question arises whether we can find solitary solutions in higher dimensions as well. In the case of scalar field theories, the following theorem due to Derrick [19] answers this question:

Consider the action for a scalar field in $D+1$ dimensions:

$$S = \int d^{D+1}x \; \frac{1}{2} F^{ab}(\boldsymbol{\phi}) \partial_\mu \phi_a \partial^\mu \phi_b - V(\boldsymbol{\phi}) \tag{C.1}$$

where $a, b = 1 \ldots N$, $F_{ab}(\boldsymbol{\phi})$ is a positive definite matrix. By adding a constant, we can always choose $V(\boldsymbol{\phi}_0) = 0$, $\boldsymbol{\phi}_0$ is the vacuum configuration of the model. The hamiltonian of the model is given by:

$$H = \int d^D x \; \frac{1}{2} F^{ab}(\boldsymbol{\phi})(\dot{\phi}_a \dot{\phi}_b + \partial_i \phi_a \partial_i \phi_b) + V(\boldsymbol{\phi}) \geq 0 \tag{C.2}$$

Let $\boldsymbol{\phi}(x)$ be a static solution of $\delta_{\boldsymbol{\phi}} H = 0$. Consider the scaled field $\boldsymbol{\phi}_\lambda(x) = \boldsymbol{\phi}(\lambda\, x)$. It is clear that $\boldsymbol{\phi}_\lambda$ should minimize (or maximize) the hamiltonian for $\lambda = 1$:[1]

$$\frac{d}{d\lambda} H[\boldsymbol{\phi}_\lambda] \mid_{\lambda=1} = 0 \tag{C.3}$$

$$\frac{d}{d\lambda} H[\boldsymbol{\phi}_\lambda] = \frac{d}{d\lambda} \int d^D x \; \frac{1}{2} F^{ab}(\boldsymbol{\phi}(\lambda x)) \partial_i \phi_a(\lambda x) \partial_i \phi_b(\lambda x) + V(\boldsymbol{\phi}(\lambda x)) \tag{C.4}$$

After a change of variable $x' = \lambda x$:

$$\frac{d}{d\lambda} H[\boldsymbol{\phi}_\lambda] = \frac{d}{d\lambda} \int d^D x' \, \lambda^{-D} \; (\lambda^2 \frac{1}{2} F^{ab}(\boldsymbol{\phi}) \partial_i \phi_a \partial_i \phi_b + V(\boldsymbol{\phi})) \tag{C.5}$$

Deriving (C.5) and setting $\lambda = 1$ together with (C.3) yields the relation:

$$(2 - D)\, F^{ab}(\boldsymbol{\phi}) \partial_i \phi_a \partial_i \phi_b = D\, V(\boldsymbol{\phi}) \tag{C.6}$$

The righthand-side of (C.6) and the term quadratic in the fields are positive. This sets the following restrictions on a static solution:

$D = 1$	$F^{ab}(\boldsymbol{\phi})\, \partial_i \phi_a\, \partial_i \phi_b = V(\boldsymbol{\phi})$	A static solutions may exist.
$D = 2$	$V(\boldsymbol{\phi}) = 0$	A static solutions may exist only in regions with zero potential.
$D > 2$	$F^{ab}(\boldsymbol{\phi})\, \partial_i \phi_a\, \partial_i \phi_b = V(\boldsymbol{\phi}) = 0$	The only static solutions are trivial, i.e. constants.

[1] Otherwise $\boldsymbol{\phi}$ would not be a static solution

The case $D = 2$ is the only interesting extension to $1 + 1$ dimensions. The condition for a nontrivial static solution is a flat potential or at least a potential with a degenerate minimum. This is what happens in the $2 + 1$ dimensional nonlinear σ-model. A static soliton exists in the massless case, but it necessarily becomes time dependent when a potential is added.

It has to be noted that this theorem applies in particular to σ-models defined on a curved internal space. The matrix $F^{ab}(\boldsymbol{\phi})$ represents the metric on this manifold.

D Equivalence of two equations for the profile function

In a series of papers [25, 26, 27], Chepilko et al. proposed a method to quantize the classically unstable soliton of the (pure[1]) Skyrme-model. As in the model considered here, there is an unknown profile function, called "chiral angle". It is argued by these authors that the equation from which the profile function F should be determined is :

$$\langle n|\frac{\delta L}{\delta F}|n\rangle = 0 \tag{D.1}$$

where L is the lagrangian (operator) and $|n\rangle$ is an energy eigenstate. An admissible profile F for a state $|n\rangle$ should make the mean lagrangian stationary. The authors warn explicitly from using another profile equation:

$$\frac{\delta E_n}{\delta F} = 0 \tag{D.2}$$

where E_n is the energy of the energy eigenstate n. They [26] say that equations (D.1) and (D.2) are not equivalent in general. It is not clear what degree of generality is required by these authors. However, it is shown in the following that the two equations are equivalent for a large class of models, including the model discussed in this work.

Let us first introduce the standard

Definition Let $L(\mathbf{q}(t), \dot{\mathbf{q}}(t), \boldsymbol{\alpha})$ be a lagrangian in terms of the dynamical variables q_i and an arbitrary number of parameters α_j. The canonical impulsions are $p_k = \partial_{\dot{q}_k} L(\mathbf{q}, \dot{\mathbf{q}}, \boldsymbol{\alpha})$ and the canonical hamiltonian is $H(\mathbf{p}, \mathbf{q}, \boldsymbol{\alpha}) = \sum_k p_k\, \dot{q}_k(\mathbf{p}, \mathbf{q}, \boldsymbol{\alpha}) - L(\mathbf{q}, \dot{\mathbf{q}}(\mathbf{p}, \mathbf{q}, \boldsymbol{\alpha}), \boldsymbol{\alpha})$. A discrete energy eigenstate $|n\rangle$ satisfies $H|n\rangle = E_n|n\rangle$.

[1] i.e. without the so-called stabilizing *Skyrme-term*.

Theorem D.1

$$\langle n|\partial_{\alpha_j} L(\mathbf{q}(t), \dot{\mathbf{q}}(t), \boldsymbol{\alpha})|n\rangle = -\partial_{\alpha_j} E_n$$

Proof Let us vary the energy with respect to the parameters α_j:

$$\partial_{\alpha_j} E_n = \partial_{\alpha_j} \frac{\langle n|H|n\rangle}{\langle n|n\rangle} = \underbrace{\frac{\delta}{\delta|n\rangle} \frac{\langle n|H|n\rangle}{\langle n|n\rangle}}_{0} \partial_{\alpha_j}|n\rangle + \langle n|\partial_{\alpha_j} H|n\rangle$$

The first term on the right hand side is the variational formulation of the eigenvalue equation $H|n\rangle = E_n|n\rangle$. It vanishes by definition of $|n\rangle$. When evaluating the remaining part, we have to be careful to retain the correct dependency on α_j:

$$\begin{aligned}
\langle n|\partial_{\alpha_j} H(\mathbf{p}, \mathbf{q}, \boldsymbol{\alpha})|n\rangle &= \langle n|\, \partial_{\alpha_j} \sum_k p_k\, \dot{q}_k(\mathbf{p}, \mathbf{q}, \boldsymbol{\alpha}) - \partial_{\alpha_j} L(\mathbf{q}, \dot{\mathbf{q}}(\mathbf{p}, \mathbf{q}, \boldsymbol{\alpha}), \boldsymbol{\alpha})|n\rangle \\
&= \langle n| \sum_k p_k\, \partial_{\alpha_j} \dot{q}_k - \sum_k \partial_{\dot{q}_k} L(\mathbf{q}, \dot{\mathbf{q}}, \boldsymbol{\alpha}) \partial_{\alpha_j} \dot{q}_k - \partial_{\alpha_j} L(\mathbf{q}, \dot{\mathbf{q}}, \boldsymbol{\alpha})|n\rangle \\
&= -\, \langle n|\partial_{\alpha_j} L(\mathbf{q}, \dot{\mathbf{q}}, \boldsymbol{\alpha})|n\rangle
\end{aligned}$$

QED.

In the limit of an infinite number of parameters α_j, the theorem shows the equivalence of (D.1) and (D.2). The possibility that some subtleties are involved in this limit can not be excluded. It should also be noted that above derivation is purely formal. No special care has been taken to handle continuous spectra etc. Therefore, the equivalence can not be regarded as fully set. However, in the model discussed in this work the equivalence has been verified explicitly.

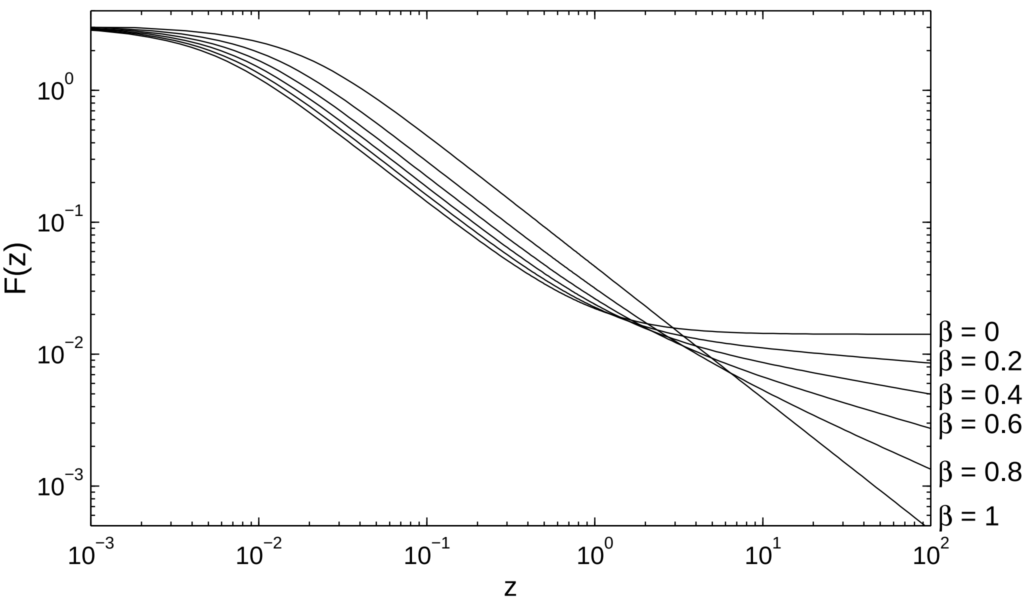

Figure E.1: The profile function for different values of the parameter β.

E Numerical procedure to solve the profile equation

In this appendix we want to explain the numerical scheme we have used to solve the profile equation (3.24). The task is to solve this equation for $\beta \in [0,1]$ with $F(z)$ smooth on $z \in (0,\infty)$ and subject to the boundary conditions $F(0) = \pi$ and $F(z) \to 0$ as $z \to \infty$.

The profile equation has a singular point at $z = 1$. We develop the solution $F(z) \simeq F(1) + F'(1)z + \ldots$ around $z = 1$. If F is smooth at $z = 1$, it has to satisfy:

$$(F' + \frac{\sin 2F}{4} + \beta\frac{\sin F}{2})|_{z=1} = 0 \tag{E.1}$$

Hence, we have two parameters $F'(0)$ and $F(1)$ which must be adjusted such that $F(z)$ is continuous and smooth on the interval $(0,1)$. It is a natural feature of singularities that one can only numerically integrate away of them. Hence, we pick an arbitrary point, $z_0 = 0.5$, and integrate the equation from $z = 0$ to $z = z_0$ and from $z = 1$ to $z = z_0$. We define

$$\Delta_1(F'(0), F(1)) = F_l(z_0) - F_r(z_0) \tag{E.2a}$$
$$\Delta_2(F'(0), F(1)) = F_l'(z_0) - F_r'(z_0) \tag{E.2b}$$

where F_l and F_r are the numerical solutions from the left respectively the

β	$B[F]$	c_2
0.0	2.0026	0.014
0.2	2.0016	0.014
0.4	2.0014	0.014
0.6	2.0013	0.015
0.8	2.0012	0.017
1.0	2.0011	0.051

Table E.1: Parameters of the found profile functions for some values of β. B is the classical energy functional and c_2 the parameter of the asymptotic solution (3.26).

right integration. The problem is solved if we find (f_1^0, f_2^0) such that $\mathbf{\Delta}(f_1^0, f_2^0) = 0$. This is a root finding problem in 2 dimensions. We have used the Newton-Raphson method with numerical derivatives to solve this problem and found solutions for any β in the desired range. The resulting profiles are shown in figure E.1. In this loglog-plot, the tail $x^{-1+\sqrt{1-\beta}}$ is apparent. The energy functional $B[F]$ and coefficients of the asymptotic solutions (3.26) are shown in table E.1.

Although $A[F]$ and $C[F]$ defined in (3.19) diverge for the solutions found, their ratio can be calculated from the asymptotics of F (3.26):

$$\begin{aligned}\frac{A}{C} &= \lim_{R\to\infty} \frac{\int^R z^3 dz\, F'^2}{2\int^R z dz\,(1-\cos F)} \\ &= \lim_{R\to\infty} \frac{\int^R z^3 dz\,(-1+\sqrt{\beta-1})^2\, c_2^2\, z^{-4-2\sqrt{\beta-1}}}{\int^R z dz\, c_2^2\, z^{-2-2\sqrt{\beta-1}}} \\ &= (-1+\sqrt{\beta-1})^2\end{aligned} \tag{E.3}$$

This result was used in (3.27).

F Numerical scheme to integrate the field equations

In the following, we want to describe briefly the numerical scheme used to integrate equation (2.30). It is a second order partial differential equation for the real function $f(x,t)$. We first write it in the form

$$\ddot{f}(x,t) = \mathcal{F}(f(x,t), f''(x,t), f'(x,t), x) \tag{F.1}$$

Space is discretized by $x_i = x_0 + i\Delta x$ and we approximate the space derivatives on the right hand side of (F.1) by

$$\begin{aligned} f'(x_i,t) &= (f(x_{i+1},t) - f(x_{i-1}))\,(2\Delta x)^{-1} + O(\Delta x)^2 \\ f''(x_i,t) &= (f(x_{i+1},t) - 2f(x_i,t) + f(x_{i-1}))\,\Delta x^{-2} + O(\Delta x)^2 \end{aligned} \tag{F.2}$$

For the time integration we use the so-called staggered leapfrog algorithm. Let the function $f(x_i, t_{n-1})$ be known at the time $t_n - \Delta t$ and the velocities $\dot{f}(x, t_{n-1/2})$ at the time $t_n - \Delta t/2$. We can approximate the field and velocities at the next time step by two calculations:

$$\begin{aligned} f(x_i,t_n) &= f(x_i,t_{n-1}) + \Delta t\,\dot{f}(x_i,t_{n-1/2}) + O(\Delta t)^2 \\ \dot{f}(x_i,t_{n+1/2}) &= \dot{f}(x_i,t_{n-1/2}) + \Delta t\,\mathcal{F}(f(x_{\{i\}},t_n),x_i) + O(\Delta t)^2 \end{aligned} \tag{F.3}$$

The core routine of the code, called *evolve()*, implements this integration. It is given below.

We actually doubled the space step Δx at large distance from the origin. This is justified by the fact that the differences in (F.2) get very small outside the soliton. The boundary conditions are simply implemented by keeping the field values fixed at $x_0 = 0$ and $x_{N_x} \simeq 5m^{-1}$.

```
int evolve(void)
{
/*  Time integration routine for 2d pde
    Samuel Bieri 04

    double f[i]         field at the point r[i]
    double fdot[i]      velocity at the point r[i]
    double fddot[i]     acceleration at r[i]
    STEP2               index at which step is doubled
    Nt, Nx              total time/space steps
*/
```

```
    int i;
    double fddot;

    for(t=0; t<Nt; t++)
    {
//Save output at certain time intervals
        if(!(t%TIMESTEP)) {
            save_data();
        }
// Advance f one time step:
        for(i=1; i< Nx-1; i++) f[i]  += dt * fdot[i];

//Advance fdot one time step
// Make small space steps first:
        for(i=1; i< STEP2; i++) {

        //Calculate fddot_i+1 as a function of f_i :
            fddot = (f[i-1]-2.*f[i]+f[i+1])/dx2 + (
                (f[i+1]-f[i-1])/dx - sin(2.*f[i])/r[i] )/(2*r[i])
                - m2*sin(f[i]);

            fdot[i] += dt * fddot;
        }

// Transition to double space step:
        fddot = (f[i-2]-2.*f[i]+f[i+1])/(4*dx2) +
            ( (f[i+1]-f[i-2])/(2.*dx) - sin(2.*f[i])/r[i] ) / (2*r[i])
            - m2*sin(f[i]);
        fdot[i] += dt * fddot;
// Continue with twice the step size:
        for(i=STEP2+1; i< Nx-1; i++) {
            fddot = (f[i-1]-2.*f[i]+f[i+1])/(4.*dx2) +
                ( (f[i+1]-f[i-1])/(2.*dx) - sin(2.*f[i])/r[i] ) / (2*r[i])
                - m2*sin(f[i]);
            fdot[i] += dt * fddot;
        }
    }
    return 0;
}
```

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